Using Mplus To Do Exploratory Factor Analysis: An Overview of the State Of The Art

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This presentation shows how to use Mplus to do exploratory factor analysis in new ways. This is joint work with Tihomir Asparouhov. The topic of EFA may seem old but there are new things to say about it. A series of Mplus techniques for EFA have recently been developed that give better results and have more general applicability. These techniques are discussed in a variety of papers and presentations that also cover many other topics, making them less easy to find. Pulling from these sources, this presentation aims to give a concise overview of the EFA state of the art using Mplus.

New techniques are presented for the traditional EFA setting of analyzing a measurement instrument for one group at one timepoint. In addition, new EFA techniques are discussed where EFA is an alternative measurement model to CFA in structural equation modeling settings, in multi-group settings, and in settings with multiple timepoints.

The technical level of this presentation is kept to a minimum with a focus on model choices, Mplus inputs, and applications to real data. The presentation assumes some basic familiarity with EFA. Mplus Short Course Topic 1 on our website gives an introduction. There are of course many books that present the basics of EFA and some are given in the reference list of this presentation - see for instance the Fabrigar-Wegener book.

Outline

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As the Outline shows, the presentation starts with a very brief background for EFA. Looking down the main bullets, the presentation is then structured as EFA variations, EFA expanded to an SEM setting, EFA for multiple groups, and EFA for multiple timepoints. This is followed by some further topics, a bit of EFA theory, and references.

Bullet 2: EFA variations considers the traditional setting of analyzing a measurement instrument for one group at one timepoint. This introduces techniques with the acronyms ESEM, PSEM, SEFA, and DSEFA. This represents a major part of the presentation.

Bullet 3: EFA in an SEM setting generalizes the traditional EFA setting to adding covariates and to replace the typical SEM CFA measurement model with an EFA measurement model.

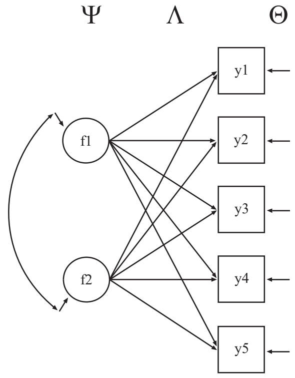
Bullet 4: EFA in a multiple-group setting presents the ESEM approach for specifying configural, metric and scalar measurement invariance across groups while still in the EFA framework, not necessitating CFA. The EFA alignment approach called AESEM meets the need for group comparisons of factor means and variances when metric and scalar invariance fits poorly, only requiring configural invariance - while still in the EFA framework.

Bullet 5: EFA in a longitudinal setting discusses an ESEM approach for metric and scalar invariance that adds between-person random intercepts and within-person factors with auto regressions. Automated longitudinal measurement invariance testing is presented. EFA longitudinal alignment fills the need for longitudinal comparisons of factor means and variances when metric and scalar invariance fits poorly. EFA growth modeling uses a PSEM approach to allow the use of an EFA measurement model when studying a growth model.

Further topics, EFA theory, and references conclude the presentation.

1 EFA background

EFA Model with Two Factors (M = 2)



- The goal of EFA is to find the smallest number of factors that explain the correlations among the observed variables
- EFA specifies only the number of factors - unlike confirmatory factor analysis (CFA), EFA has no hypothesis about zero loadings
- The EFA model is not identified but has m^2 indeterminacies
- The EFA model is made identified by applying a "rotation" that eliminates the m^2 indeterminacies and gives an interpretable model

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Slide 3 shows an EFA model for 6 factor indicators measuring 2 factors. The single-headed arrows represent linear regressions when indicators are continuous and probit or logit regressions for categorical indicators.

The goal of EFA is to find the smallest number of factors that explain the correlations among the observed variables.

EFA specifies only the number of factors, no left-out arrows from the factors to the indicators as in CFA. Characteristic of EFA, both factors influence all 6 indicators.

As stated in the third bullet, the EFA model is not identified but has $m^2 = 4$ indeterminacies. The correlations among the 6 indicators is expressed by 3 parameters in the factor covariance matrix Ψ - the covariance and the two variances of the factors - as well as $2 \times 6 = 12$ parameters in the factor loading matrix Λ, for a total of 15 parameters. Although there are also 15 sample correlations, the model is not identified, that is, there is no unique solution for the parameter estimates but there are indeterminacies. For example, a change in the factor variance can be compensated by a change in the loadings resulting in the same variances and covariances for the indicators.

As stated in the last bullet, the indeterminacies can be removed by EFA rotation. This solves the problem of indeterminacies and presents an opportunity to explore a solution that is clearly interpretable.

As a start, an unrotated model is created where the factor variances are standardized to 1 and the factor covariance is set at 0 (referred to as orthogonal rotation; see EFA theory section). This leaves 1 remaining indeterminacy which can be settled by fixing a loading to zero. Based on this unrotated solution, the rotation then chooses a simple and more interpretable set of factor loading values where simple means that an indicator should have one or two large loadings in a row of the loading matrix to help define the factor but also small loadings for other factors. The rotation can also relax the zero factor correlation (referred to as an oblique solution) to further simplify the loading pattern. Rotation for two factors is illustrated graphically for two examples in the EFA theory section.

Stages of Measurement Development Using Factor Analysis

Stage	Specification	Technique
Early	# of factors only	EFA, ESEM PSEM with GEOMIN priors
Middle	Pre-specified zero loadings Pre-specified key loadings	Target rotation Cross-loadings constrained using PSEM ALF priors
Late	# of factors, key loadings, and zero cross-loadings	CFA Modification indices

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Turning to slide 4, it is useful to distinguish between different stages of measurement development using factor analysis: Early, Middle, and Late.

At an early stage, relatively little is known about how well factor indicators are able to capture hypothesized factors. You may be ready to specify only how many factors the indicators measure or perhaps only a range of number of factors. At this stage, traditional EFA is the technique of choice. Automatic rotation procedures search for a simple, interpretable factor loading matrix. Indicators shown to be poor measures of factors are either reformulated or deleted. For a well-defined EFA model, the techniques of ESEM and PSEM make it possible to move to more general analysis settings.

At the middle stage, experimentation with choices of factor indicators makes it possible to specify not only the number of factors but also which indicators do not load on certain factors, leading to the technique of target rotation, that is, a rotation that is guided by prior knowledge rather than automatic criteria of simplicity. Alternatively, it is possible to specify which indicators are key to measuring which factors, leading to the technique of studying cross loadings using PSEM ALF priors.

At the late stage, it may be possible to specify the number of factors, key loadings, as well as zero cross loadings, leading to a model with more restrictions than EFA. This leads to the technique of CFA (confirmatory factor analysis; see Jöreskog's 1969 Psychometrika article) which is geared towards testing specific factor structures. Model misfit is captured by modification indices.

Two comments on these three stages are of interest.

First, a common problem of factor analysis is that the late stage approach of CFA is often the first approach used, skipping the two earlier stages. This typically happens in the context of testing a hypothesis that has been well established in the literature - except not for the indicators or population at hand. As a consequence, poor fit is obtained and a long search for better fit via modification indices is carried out. As argued by Michael Brown (2001) quoted in the Asparouhov & Muthén (2009) ESEM article, EFA is in this case a better alternative where rotation is a more direct approach to finding misfitting cross loadings than using modification indices.

Second, in many applications, CFA is not a possible or desirable late stage analysis. The CFA model may not fit well despite modifications. Due to the nature of what is measured, it is often not reasonable to expect that relevant factor indicators measure only one factor but have loadings on other factors, albeit with smaller values. The factor loading pattern is then more complex and better suited to an EFA model. In this situation, staying with an ill-fitting CFA model tends to inflate the factor correlations. In the past, an ill-fitting CFA was a hindrance to further analyses in settings such as structural equation modeling, multiple groups, and multiple timepoints. With the ESEM and PSEM advances shown in this presentation, however, it is possible in all these settings to replace the CFA measurement model with an EFA counterpart. These arguments are further discussed in Asparouhov & Muthén (2009) as well as in Muthén & Asparouhov (2012).

Technical Terms Used in the Talk

- Rotation
  - A way to identify the model and get interpretable factor loadings
    - GEOMIN and many others, orthogonal or oblique (uncorrelated vs correlated factors), target, bi-factor (see EFA Theory section)
- ESEM: Exploratory structural equation modeling
  - An approach to do use EFA instead of CFA measurement models in an SEM context
- PSEM: Penalized Structural Equation Modeling
  - A rotation approach and specification of small effects using priors
- Priors
  - A way to restrict the size of a parameter or sets of parameters
  - Common with Bayes estimation but here used with PSEM for maximum-likelihood and WLSMV estimation
    - GEOMIN priors for loadings with rotation, ALF (alignment loss function) for small effects
- AESEM
  - ESEM alignment of measurements across groups or timepoints

Slide 5 gives a summary of terms that will be used throughout the presentation.

Bullet 1: We've already touched on rotation as a way to make the EFA model identified and to obtain interpretable factor loadings. GEOMIN is the Mplus default rotation and is discussed in the 2009 Asparouhov-Muthén ESEM article. Both orthogonal and oblique rotations are available as listed in the Mplus User's Guide. Graphical representation of rotation examples are shown in the EFA Theory section of the presentation.

Bullet 2: ESEM - exploratory SEM - will be featured prominently in this presentation and its main features are described in the 2009 Asparouhov-Muthén article with several more recent additions. The main advantage is to be able to do SEM with an EFA measurement model instead of CFA.

Bullet 3: PSEM - penalized SEM - has two main uses. One is to do EFA rotations with GEOMIN also in non-standard models such as second-order factor analysis. The more general use is with effects hypothesized to be small using ALF priors.

Bullet 4: Using priors is a way to restrict the size of parameters. PSEM priors are used with maximum-likelihood or - with categorical indicators - WLSMV estimation. GEOMIN priors are used for rotating loading matrices. ALF (Alignment Loss Function) priors are used for small effects.

Bullet 5: AESEM stands for alignment using ESEM. Alignment refers to measurement invariance assuming only configural invariance across groups or timepoints.

2 EFA variations

Outline

- EFA background
- **EFA variations**
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- EFA theory

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Slide 6 returns to the Outline of the presentation and we turn to the topic of EFA Variations. Here we are going to get into applications of ESEM and PSEM. We will also discuss the new Mplus technique of exploratory second-order factor analysis, abbreviated as SEFA. Exploratory bi-factor analysis will also be covered, including the new Mplus technique of DSEFA - direct effects second-order factor analysis.

2.1 Hypothesis about the number of factors

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors
  - Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM
- Hypothesis about the number of factors and key items:
  - ESEM with Target rotation
  - PSEM with ALF priors for cross loadings
- Comparing EFA methods
- Special models:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings

The EFA Variations topic is a major part of this presentation and is longer than the other parts. Because of this, slide 7 unpacks this topic, showing its parts. Here, the first two main bullets make a distinction between a hypothesis about the number of factors and hypotheses that also include key items. This distinction is in line with the slide 4 discussion of the early and middle stages of factor analysis.

For the main bullet 1, the first 3 sub bullets shows 3 ways that regular EFA can be carried out in Mplus: TYPE = EFA, ESEM, and PSEM GEOMIN. Following this, exploratory versions of second-order and bi-factor analysis using PSEM are discussed.

Main bullet 2 covers cases where key items are specified. Here, two main techniques will be described: ESEM with target rotation, and PSEM ALF priors for cross-loadings.

Main bullet 3 compares EFA methods with a focus on exploratory second-order and bi-factor analyses, but also considers target-oriented approaches.

The EFA Variation section ends with a brief discussion of 2 special models.

2.1.1 Three Regular EFA approaches. H&S example

Regular EFA Using Three Mplus Approaches: Analysis of Two Examples

- TYPE = EFA
- ESEM (*1)
  - Asparouhov & Muthén (2009). Exploratory structural equation modeling. Structural Equation Modeling. 16, 397-438.
- PSEM with GEOMIN priors
  - Asparouhov & Muthén (2024). Penalized structural equation models. Structural Equation Modeling. 31, 429-454.
  - Asparouhov & Muthén (2025). Methodological advances with penalized structural equation models. Structural Equation Modeling. 32, 688-716.

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We now turn to two examples. Slide 8 shows 3 Mplus approaches to EFA that we will use.

TYPE = EFA allows the analysis of a range of factors in one analysis, computing chi-square difference tests between number of factors.

The ESEM approach gives an equivalent approach for a specific number of factors and also serves as a bridge towards more general EFA settings to be discussed later. The 2009 article is the seminal source of the theory for ESEM.

The PSEM approach gives another equivalent approach using the GEOMIN priors that are the default for the TYPE = EFA approach. As shown in the two articles, this also serves as a bridge towards more general EFA settings to be discussed later.

Holzinger & Swineford (1937, 1939) Four Domains Measured by 19 Tests: Factor Loading Pattern

Test	Spatial	Verbal	Speed	Memory
Visual perception	X	0	0	0
Cubes	X	0	0	0
Paper form board	X	0	0	0
Flags	X	0	0	0
General information	0	X	0	0
Paragraph comprehension	0	X	0	0
Sentence completion	0	X	0	0
Word classification	0	X	0	0
Word meaning	0	X	0	0
Addition	0	0	X	0
Code	0	0	X	0
Counting groups of dots	0	0	X	0
Straight and curved capitals	0	0	X	0
Word recognition	0	0	0	X
Number recognition	0	0	0	X
Figure recognition	0	0	0	X
Object–number	0	0	0	X
Number–figure	0	0	0	X
Figure–word	0	0	0	X

On slide 9 we have our first example. This is the classic dataset used in the 1937 and 1939 Holzinger-Swineford papers given in the references. The figure shows a loading matrix for 19 tests with X marking hypothesized large loadings and 0 elsewhere. Four factors are measured by these tests: Spatial, Verbal, Speed, and Memory. Later on we will add 5 more tests and look at the bi-factor model of primary interest to Holzinger and Swineford.

In terms of analysis stages discussed on slide 4, having this hypothesized loading pattern corresponds to the late analysis stage of CFA but here we will primarily focus on EFA approaches. The dataset is available on the web paper page.

H&S Model with 4-factor CFA Using MLR

- Seventh- and eighth-grade students from two schools
- Grant–White (N=145): students from homes where the parents were mostly American born
- Pasteur (N = 156): students largely from working-class parents of whom many were foreign born and used their native language at home

Grant-White School

Model	Par's	LL	BIC	χ^2	Df	P	CFI
CFA	63	-9050	18412	217	146	0.000	0.927
Pasteur School							
CFA	63	-9909	20137	273	146	0.000	0.871

On slide 10, we will first show results from a 4-factor CFA using the loading pattern of the previous slide.

The dataset consists of 7th and 8th grade students from two schools, Grant-White and Pasteur. Each has a sample of about 150 students. As the table shows, both schools get a poorly fitting model as judged by chi-square with zero p-value. Given the moderate sample size, the poor fit cannot be ascribed to rejection due to a large sample. Also supporting the rejection, the MLR estimator is used which reduces the sensitivity of the chi-square test to nonnormality in the factor indicators.

CFA MLR Modification Indices for H&S Pasteur School

	M.I.	E.P.C.	Std E.P.C.	StdYX E.P.C.
BY Statements				
SPATIAL BY ADDITION	10.263	-0.301	-0.301	-0.301
SPATIAL BY FIGURER	13.969	0.344	0.344	0.344
VERBAL BY VISUAL	11.835	0.402	0.402	0.402
SPEED BY NUMBERR	12.957	-0.361	-0.361	-0.361

- Freeing the 4 cross-loadings still gives poor chi-square with zero p-value: Chi-2 (142) = 222
- A second round of freeing large modindices can be done but EFA may be a better alternative, giving results in one run
- See arguments in favor of EFA in the ESEM paper Asparouhov & Muthén (2009)

Slide 11 shows the key modification indices for the CFA run corresponding to unexpected cross loadings. The expected parameter change values for these are rather large in the standardized scale so these are substantively important misspecifications.

As the first bullet says, however, freeing these 4 cross loadings still results in poor fit. Instead of going further as guided by modification indices, EFA is a better approach in line with arguments mentioned on slide 4 in the ESEM paper quote of Brown.

TYPE = EFA Input for H&S

TITLE: Raw data from Holzinger-Swineford's monograph Grades VII and VIII of the Pasteur elementary school (n=156)
followed by Grant-White (n=145).
Gender, grade, and age information, and the 24 tests (Pasteur does not have tests 25 and 26)
Sources: Holzinger, K. J. & Swineford, F. (1939). A study in Factor
Analysis. The Stability of a Bi-Factor Solution. Supplementary Educational Monographs. Chicago, Ill.: The
University of Chicago.
Harman, H.H. (1976). Modern Factor Analysis. Third Edition. Chicago: The University of Chicago Press

DATA: FILE IS H-S Combined.txt;

VARIABLE: NAMES = names are id female grade agey agem school
visual cubes paper flags general
paragrap sentence wordc wordm addition code counting straight wordr figurer object numberf
figurew deduct numeric
problemr series arithmet;
USEVARIABLES =
visual-figurew;
USEOBSERVATIONS =
school eq 0; !school = 0 for
! Grant-White, 1 for Pasteur

ANALYSIS: **TYPE = EFA 3 5**;
! GEOMIN is the default rotation
ESTIMATOR = MLR;
! STARTS = 20;

PLOT: TYPE = PLOT3;

Slide 12 shows the input for EFA where in the ANALYSIS command, EFA with 3, 4, and 5 factors is requested. GEOMIN is the default rotation.

For some special situations, it may be appropriate to add several random starting values using the STARTS option. Here, 20 such sets of starting values are used.

ESEM Input for H&S

ANALYSIS: ESTIMATOR = MLR;

MODEL: **f1-f4 BY visual-figurew(*1)**;

- ESEM is activated by (*1), which designates an EFA "block" of items that are indicators of the set of factors listed.
  The "1" is arbitrary and can be replaced with e.g. "a" or "efa"
- The ESEM approach to EFA allows the usual choices of rotations
- ESEM allows more general modeling including residual covariances and including other variables such as covariates
- ESEM generalizes to SEM settings where there can be more than one EFA block and where EFA blocks can be combined with CFA blocks

Slide 13 shows the ESEM approach for 4 factors. ESEM is activated by (*1), which designates an EFA "block" of items that are indicators of the set of factors listed, such as f1-f4 BY visual-figurew. The "1" is arbitrary and can be replaced with e.g. "a" or "efa".

The ESEM approach to EFA allows the usual choices of rotations: orthogonal or oblique as well as bifactor.

The ESEM specification allows more general modeling including residual covariances and including other variables such as covariates. In fact, ESEM generalizes to SEM settings where there can be more than one EFA block and where EFA blocks can be combined with CFA blocks.

PSEM with GEOMIN Priors Input for H&S

```
ANALYSIS:       ESTIMATOR = MLR;
                ! some of the below settings might
                ! be needed:
                ! ITERATIONS = 10000;
                ! CONVERGENCE = 0.000001;
                ! STARTS = 50;

MODEL:          ! label the 4*19=76 factor loadings
                ! for which the GEOMIN rotation
                ! should be applied:
                f1-f4 BY visual-figurew*(a1-a76);
                f1-f4@1;

MODEL PRIORS:   a1-a76 ~ GEOMIN(4,1.0); ! GEOMIN settings
                ! are shown at the end of the Theory section
```

- The GEOMIN priors are applied to all factor loadings which gives the same rotated solution as the default for TYPE = EFA with 4 factors
- It is recommended to standardize the factor indicators before analysis so that the priors work optimally. This can be done using the STANDARDIZE = visual-figurew; option of the DEFINE command.

Slide 14 shows the input for PSEM with GEOMIN priors for the H&S data. This is our first PSEM example in that it uses priors. The commented lines explain the specifications.

In the MODEL command, parameter labels are given to all 76 factor loadings.

The MODEL PRIORS command specifies GEOMIN priors for the loadings. This gives essentially the same rotated solution as the default GEOMIN rotation for TYPE = EFA with 4 factors.

The GEOMIN setting “4” refers to the number of factors and 1.0 refers to the prior variance or rather 1/the penalty weight. GEOMIN settings are given in the EFA Theory section.

The factor indicators in the Holzinger-Swineford data have very different variances. We recommend standardizing the factor indicators before analysis so that the priors work optimally. This can be done using the STANDARDIZE option in the DEFINE command. For EFA, the standardization does not affect the chi-square testing.

As we will see, the GEOMIN priors approach is very convenient in that it can be used for other applications such as second-order analysis, bi-factor analysis, EFA in structural equation modeling settings, and EFA in longitudinal settings.

Summary of Model Fit Information using TYPE = EFA 3 5 for Grant-White with MLR

Model	# Parameters	Chi-Square	Df	P-Value
3-factor	92	207.070	117	0.0000
4-factor	108	116.071	101	0.1450
5-factor	123	98.474	86	0.1688

Models Compared	Chi-Square	Df	P-Value
3-factor against 4-factor	117.689	16	0.0000
4-factor against 5-factor	17.428	15	0.2939

- Choose the 4-factor model
- Chi-square difference testing with MLR uses the special approach described in `https://www.statmodel.com/chidiff.shtml`
- Asparouhov & Muthén (2024b) discusses alternative tests of the number of factors

Slide 15 shows the chi-square test of fit for 3, 4, and 5 factors. This is for the Grant-White school using MLR. Both a 4- and a 5-factor model fit well, but chi-square difference testing shows that 4 factors is sufficient in that it cannot be rejected when tested against 5 factors. So, we choose 4 factors.

With MLR, the chi-square difference testing uses the Satorra-Bentler approach that you find described on the Mplus website:
`https://www.statmodel.com/chidiff.shtml`

Alternative ways to test the number of factors in EFA is discussed in the Asparouhov & Muthén (2024) web note 25.

H&S Models with 4-factor CFA and EFA Using MLR

Model	Par's	LL	BIC	χ^2	Df	P	CFI	X-loads
Grant-White School, N = 145								
CFA	63	-9050	18412	217	146	0.000	0.927	-
EFA	108	-8997	18532	116	101	0.145	0.985	5
CFA + 5	68	-9025	**18388**	167	141	0.068	0.974	
Pasteur School, N = 156								
CFA	63	-9909	20137	273	146	0.000	0.871	-
EFA	108	-9843	20232	136	101	0.011	0.964	8
CFA + 8	71	-9872	**20103**	193	138	0.001	0.944	

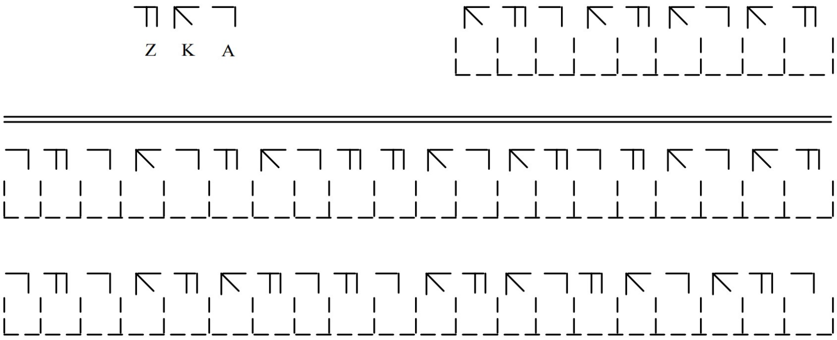
- X-loads refers to number of significant cross-loadings
- BIC = -2*LL + penalty, where penalty = # parameters * ln (N)

Slide 16 shows CFA and EFA test results for both schools. For both schools, EFA fits better in that its chi-square test gets a non-zero p-value unlike CFA. BIC, however, points to CFA. The BIC formula is shown at the bottom of the slide. This disagreement is not uncommon. In a sense, BIC shortchanges EFA. Not all EFA cross loadings are expected to be significant but these extra parameters are counted in the BIC penalty. As the table shows, EFA has a better log likelihood (LL) but many more parameters. The log likelihood improvement for EFA is not enough to compensate for the BIC penalty due to the large increase in number of parameters.

Out of the total of 57 cross loadings, Grant-White EFA has 5 that are significant and Pasteur has 8. Freeing these cross loadings in a modified CFA, the table shows that the best BIC is obtained for both schools. In this way, EFA provides an alternative to modification indices in terms of modifying a CFA. EFA may be closer to a well-fitting model than CFA so modifications of CFA based on EFA can be more likely to succeed than modifying the original CFA model by modification indices.

Examining Cross Loadings

- Code is an indicator of the Speed factor but also has a significant positive loading on the Memory factor for Grant-White
  - Requires matching letters to a set of figures



Slide 17 shows an example of a cross loading in the analysis of H&S. This is one of the 5 significant cross loadings from the EFA of Grant-White. The indicator called code measures the speed factor but also loads on the memory factor. This makes sense in that the icons for the Z, K, A letters have to be remembered when quickly filling in the letters under the icons.

The cross loading is added to the CFA by saying: memory BY code;

2.1.2 British Household Panel example. Correlated residuals

Example: British Household Panel Data (N = 589) EFA with Correlated Residuals

Wording and Hypothesized Factor Loading Pattern for the 15 Items Used to Measure the Big Five Personality Factors in the British Household Panel Data ("I See Myself As Someone Who...")

Item	A	C	E	N	O
y1: Is sometimes rude to others (reverse scored)	X	0	0	0	0
y2: Has a forgiving nature	X	0	0	0	0
y3: Is considerate and kind to almost everyone	X	0	0	0	0
y4: Does a thorough job	0	X	0	0	0
y5: Tends to be lazy (reverse scored)	0	X	0	0	0
y6: Does things efficiently	0	X	0	0	0
y7: Is talkative	0	0	X	0	0
y8: Is outgoing, sociable	0	0	X	0	0
y9: Is reserved (reverse scored)	0	0	X	0	0
y10: Worries a lot	0	0	0	X	0
y11: Gets nervous easily	0	0	0	X	0
y12: Is relaxed, handles stress well (reverse scored)	0	0	0	X	0
y13: Is original, comes up with new ideas	0	0	0	0	X
y14: Values artistic, aesthetic experiences	0	0	0	0	X
y15: Has an active imagination	0	0	0	0	X

Note. A = Agreeableness; C = Conscientiousness; E = Extraversion; N = Neuroticism; O = Openness.

Slide 18 shows a different example using data from the British Household Panel. A simple loading pattern for 5 factors is hypothesized. The factor names are listed below the table. This measurement instrument has 4 indicators that have negative statements as opposed to all the other indicators so they are reverse scored.

This example illustrates EFA with correlated residuals using ESEM.

EFA and ESEM Inputs

EFA

```
ANALYSIS:
        TYPE = EFA 5 5;
```

ESEM

```
MODEL:
        f1-f5 BY y1-y15 (*1);
```

ESEM with residual covariances

```
MODEL:
        f1-f5 BY y1-y15 (*1);
        y1 y5 y9 y12 WITH y1 y5 y9 y12;
```

- 6 residual covariances for the reverse-scored items y1, y5, y9, y12 (Marsh et al., 2013)

Slide 19 shows EFA and ESEM inputs for this dataset. The bottom input shows the strength of ESEM in that it can include residual covariances in the model. These residual covariances are added for the reverse scored indicators per suggestion of the paper by Marsh et al. (2013) which analyzed this dataset. In this way, negatively worded indicators are allowed to correlate beyond what the factors would predict.

BHP 5-Factor Solutions Using MLR

Model	# par's	LL	BIC	χ^2	Df	P	CFI
1. CFA	55	-11647	23645	415	80	0	0.78
2. CFA + 6 rescovs	61	-11610	23609	348	74	0	0.82
3. EFA	95	-11446	23498	101	40	0	0.96
4. EFA + 6 rescovs	101	-11430	23504	80	34	0	0.97

- CFA models 1 and 2 are outperformed by EFA models 3 and 4 as judged by BIC
- EFA model 3 does not capture hypothesized F1 and F2 factors
- EFA + rescovs model 4 needed which can be done by ESEM (input on previous slide)

Slide 20 shows the model fit for 4 different models. Comparing the CFA models 1 and 2, it is seen that the addition of the 6 residual covariances is warranted given better BIC and CFI. The CFA models are however outperformed by the two EFA models in terms of BIC. The EFA models also have better CFI. EFA model 3 has the best BIC but it doesn't capture the first two factors. Not adding the residual covariances throws off the modeling. EFA model 4 which is done by ESEM gives a BIC value only slightly worse than for model 3 and is the model of choice.

2.1.3 Second-order EFA (SEFA) using PSEM with GEOMIN priors

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - **Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors**
  - Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM
- Hypothesis about the number of factors and key items:
  - ESEM with Target rotation
  - PSEM with ALF priors for cross loadings
- Comparing EFA methods
- Special models:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings

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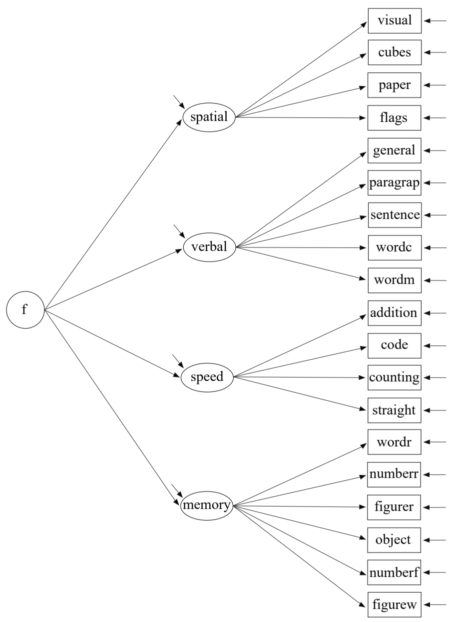
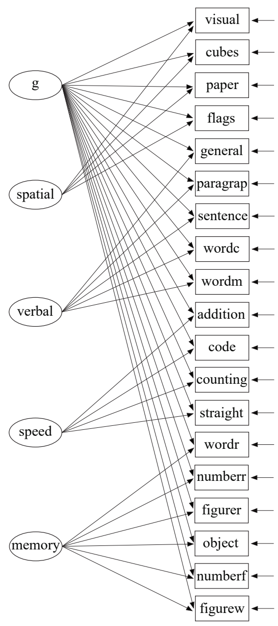
Slide 21 returns to the overview of EFA Variations. We now turn to the topic of second-order factor analysis where we will also introduce the SEFA model using PSEM with GEOMIN priors.

The SEFA model also provides the basis for the new DSEFA approach to bi-factor modeling.

Second-Order and Bi-Factor Analysis





Second-order CFA for HS19 | Bi-factor CFA for HS19

Slide 22 shows model diagrams for confirmatory factor analysis of a second-order factor model on the left and a bi-factor model on the right. The diagrams use the example of the H&S data discussed earlier with 19 tests used as factor indicators. Each of the models feature the 4 factors discussed earlier: spatial, verbal, speed, and memory. But the 4 factors play different roles in the two models. Both models consider if there is some sort of overall ability dimension that influences the performance on the tests. This could be a useful summary of ability either in its own right or in relations to background variables or prediction of future outcomes. The two models formulate this overall ability in different ways.

For the second-order model on the left, there is second-order factor f which influences the 4 first-order factors and therefore indirectly influences the indicators. The factor f is what all of the first-order factors have in common and it fully explains the correlations among the first-order factors.

For the bi-factor model on the right, there is a general factor that directly influences all the indicators. In addition, there are 4 specific factors that can be seen as residual factors, explaining correlations among the indicators beyond what the general factor can explain. All 5 factors in this model are typically specified to be uncorrelated.

Input for Second-Order Confirmatory Factor Analysis Using 4 Factors for H&S with 19 Indicators

```
ANALYSIS:    ESTIMATOR = MLR;

MODEL:       spatial BY visual-flags*;
             verbal BY general-wordm*;
             speed BY addition-straight*;
             memory BY wordr-figurew*;
             spatial-memory@1;
             ! specify a second-order factor with
             ! the first-order factors as indicators:
             f BY spatial-memory*; f@1;
```

- CFA model fit with 4 factors, both schools (N=301) using MLR:
  Chi-square (148) = 320, p = 0.000

Slide 23 shows the input for the second-order CFA model on the previous slide. All loadings are allowed to be free. The metric of the factors is set by fixing the factor variances at 1.

The CFA chi-square test of fit is shown at the bottom and indicates that the model fits poorly.

Exploratory Second-Order Factor Analysis (SEFA)

Asparouhov & Muthén (2026): A Unification of Second-Order and Bi-Factor EFA

- Exploratory EFA for first-order factors using Geomin rotation of their loadings like in regular EFA mentioned earlier
- Single second-order factor with first-order factors as factor indicators thereby summarizing the first-order factors succinctly
- Same model fit as EFA with the same number of first-order factors - just another type of rotation
  - The second-order factor imposes restrictions on the first-order factor correlations
  - This is compensated by the factor loadings rotation to give the same fit as without these factor correlation restrictions
  - This means that the rotated factor loadings for the second-order EFA model are slightly different from those of regular EFA
- The second-order factor can be related to other variables such as covariates which is a great advantage in terms of parsimony
- Available also for categorical indicators using WLSMV estimation

Slide 24 turns to exploratory second-order factor analysis, referred to as SEFA. This was introduced in our 2026 paper, A unification of second-order and bi-factor EFA.

SEFA is an example of PSEM analysis. The bullets show the features of SEFA.

Exploratory EFA for first-order factors uses Geomin rotation of their loadings like in regular EFA mentioned earlier.

There is a single second-order factor with first-order factors as factor indicators thereby summarizing the first-order factors succinctly.

SEFA has the same model fit as EFA with the same number of first-order factors; SEFA is just another type of rotation where:

- The second-order factor imposes restrictions on the first-order factor correlations.

- This is compensated by the factor loadings rotation to give the same fit as without these factor correlation restrictions.

- This means that the rotated factor loadings for the second-order EFA model are slightly different from those of regular EFA.

The second-order factor can be related to other variables such as covariates which is a great advantage in terms of parsimony.

Pre-standardization of indicators important with widely varying variances so that the priors are used optimally.

SEFA is available not only for maximum-likelihood estimation of continuous indicators but also for categorical indicators using WLSMV estimation.

EFA Tests of Model Fit with 4 and 5 Factors, Both Schools, MLR

SUMMARY OF MODEL FIT INFORMATION

Model	Number of Parameters	Chi-Square	Degrees of Freedom	P-Value
4-factor	108	131.890	101	0.0212
5-factor	123	112.680	86	0.0284

Models Compared	Chi-Square	Degrees of Freedom	P-Value
4-factor against 5-factor	19.369	15	0.1975

- 4 factors appears preferrable and agrees with hypotheses

Slide 25 shows tests of model fit for EFA. We have seen this type of EFA output before but now it is presented for both schools (N = 301). These EFA results are shown because as stated on the previous slide, EFA has the same model fit as the SEFA exploratory second-order factor analysis model with the same number of first-order factors. SEFA simply results in a different rotation, a rotation that assumes a second-order factor. It is seen that 4 factors is preferable also for the combined sample.

Input for Second-Order Exploratory Factor Analysis of H&S 19: SEFA with Geomin Priors

```
ANALYSIS:            ESTIMATOR = MLR;
                     ! some of the following settings
                     ! are sometimes needed:
                     ITERATIONS = 10000;
                     CONVERGENCE = 0.000001;
                     STARTS = 50;

MODEL:               ! do a GEOMIN rotation on the
                     ! 4*19=76 first-order factor loadings:
                     f1-f4 BY visual-figurew*(a1-a76);
                     ! specify a second-order factor:
                     f0 BY f1-f4* (l1-l4); f0@1;
                     ! restrict the residual variances so total
                     ! variances are 1 for the first-order factors:
                     f1-f4 (v1-v4);

MODEL CONSTRAINT:    v1 = 1 - l1*l1;
                     v2 = 1 - l2*l2;
                     v3 = 1 - l3*l3;
                     v4 = 1 - l4*l4;

MODEL PRIORS:        a1-a76 ~ GEOMIN(4,0.1);
```


Slide 26 shows the input for SEFA as applied to the H&S dataset with 4 first-order factors. This is what we call the manual specification of the model. In Mplus Version 9.1 the input is to a large extent simplified by automatically doing most of the setup. This simplified input will also be shown.

The commented lines explain the input for the manual specification. Importantly, the GEOMIN rotation is applied to the first-order factors as seen in the labeling of the loadings and also in the MODEL PRIORS command. As pointed out in the paper, the residual variances need to be restricted so the total variance of each first-order factors is 1. This is carried out by the MODEL CONSTRAINT command.

Input for EFA Starting Values for Second-Order Exploratory Factor Analysis of H&S 19

- EFA starting values for the f1-f4 factor loadings can be conveniently obtained via ESEM where the loadings are labeled for use in the SEFA
- This typically avoids using the settings for number of iterations, convergence, and starts
- ESEM input:

```
MODEL:   f1-f4 by visual-figurew(*1);
         f1-f4 by visual-figurew(a1-a96);

OUTPUT:  SVALUES;
```

- Copy the estimated values for the f1-f4 loadings from the output and paste them in the SEFA input

Slide 27 shows the input for getting SEFA starting values using EFA in ESEM form. This is not needed for the simplified input but is shown here for completeness.

Starting values are helpful for avoiding the ANALYSIS specifications on the previous slide for iterations, convergence, and starts.

The ESEM input gives labels for the loadings which will be printed in the output by requesting SVALUES.

The SVALUES output for the loadings is then used in the SEFA input.

Simplified SEFA Input

- Automated setup of model and priors
  - Special rotation
  - ESEM specification

```
ANALYSIS:   ESTIMATOR = MLR;
            ROTATION = SEFA;
            ! SEFA settings shown at the end
            ! of the Theory section

MODEL:      f1-f4 BY visual-figurew(*1);
            f BY f1-f4;
            ! covariates can be added
```

- Pre-standardization of factor indicators recommended
- EFA-generated starting values
- SEFA-specific output
- Current limitations
  - Single group, at least 3 first-order factors, single EFA block

Slide 28 shows the simplified, automated input for SEFA replacing the previous two slides. The automated approach is activated by the ROTATION = SEFA statement, emphasizing that SEFA implies a specific rotation. Special settings can be given in parentheses, such as SEFA (0.1) corresponding to the GEOMIN setting of slide 26. The settings are described in the EFA Theory section.

The first-order factors are defined using ESEM style input. The added second-order factor is specified in usual style. Covariates can be added.

Standardization of the indicators is recommended when the indicators have widely varying variances.

The analysis uses starting values from an EFA analysis with the same number of factors as the number of first-order factors in the SEFA.

The automated approach to SEFA is currently limited to a single group with at least 3 first-order factors, as well as a single EFA block for ESEM.

SEFA Results for H&S19 Both Schools (N = 301), MLR

ROTATED LOADINGS (* significant at 5% level)

	Spatial	Verbal	Speed	Memory
VISUAL	**0.621***	0.155*	0.024	0.049
CUBES	**0.514***	0.048	-0.110	-0.021
PAPER	**0.465***	0.099	0.006	-0.070
FLAGS	**0.632***	-0.091	0.026	0.110
GENERAL	-0.011	**0.846***	0.040	-0.082
PARAGRAP	0.014	**0.802***	-0.006	0.068
SENTENCE	-0.049	**0.908***	-0.008	-0.059
WORDC	0.081	**0.697***	0.022	0.039
WORDM	0.072	**0.820***	-0.033	0.026
ADDITION	-0.218*	0.014	**0.764***	0.062
CODE	0.031	0.174*	**0.542***	0.161*
COUNTING	0.108	-0.032	**0.674***	-0.070
STRAIGHT	0.355*	0.010	**0.497***	-0.030
WORDR	-0.044	0.075	-0.027	**0.653***
NUMBERR	0.081	-0.122*	-0.004	**0.586***
FIGURER	0.317*	0.045	0.014	**0.449***
OBJECT	-0.141	-0.037	0.334*	**0.534***
NUMBERF	0.090	0.011	0.188*	**0.401***
FIGUREW	0.081	0.174*	0.060	**0.305***

F0 BY	
Spatial	0.620*
Verbal	0.538*
Speed	0.503*
Memory	0.431*

FACTOR CORRELATIONS (* significant at 5% level)

	Spatial	Verbal	Speed	Memory
Spatial	1.000			
Verbal	0.333*	1.000		
Speed	0.312*	0.271*	1.000	
Memory	0.267*	0.232*	0.217*	1.000
F0	0.620*	0.538*	0.503*	0.431*

.

Slide 29 shows the estimated factor loadings for the 4 first-order factors Spatial, Verbal, Speed, and Memory. All factors have large, significant loadings in the pattern that was hypothesized.

Bottom left shows that the first-order factors all have large, significant loadings on the second-order factor with largest loading for Spatial and smallest for Memory. The correlations among all 5 factors is also shown. We see that the first-order factors have moderate correlations. Of course, if they have small correlations, it doesn't make much sense to do SEFA.

As a footnote, the Spatial - Memory labeling of the columns of the factor loading matrix on slide 29 is not provided by Mplus but done afterwards based on interpretation. The Mplus output gives the f1-f4 labeling used in the input.

Comparison with Regular EFA 4-Factor Oblique Solution

- Same model fit as regular 4-factor EFA
- For this example, the regular 4-factor EFA with the default oblique roations has similar factor loadings and factor correlations
  - Same number of significant cross-loadings and cross-loadings larger than 0.2 in absolute value
- The second-order factor can be related to other variables such as covariates

Slide 30 reminds us that the SEFA model has the same fit as a regular 4-factor EFA.

The default oblique rotation of EFA has similar factors loadings and factor correlations to those of SEFA. It has the same number of significant cross loadings and cross loadings larger than 0.2. This means that nothing is lost in terms of simplicity relative to regular EFA, but what is gained is the second-order factor summary of abilities.

The second-order factor can then be related to background variables and be used for predicting other outcomes.

2.1.4 Bi-factor analysis using ROTATION=BI-GEOMIN and direct effects second-order EFA (DSEFA) using PSEM

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors
  - **Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM**
- Hypothesis about the number of factors and key items:
  - ESEM with Target rotation
  - PSEM with ALF priors for cross loadings
- Comparing EFA methods
- Special models:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings
  - Mixture EFA
  - Twolevel EFA

Slide 31 once again shows the outline of the various EFA Variations. We now turn to bi-factor analysis. Both old and new, improved techniques are discussed.

The focus is on the new method of direct second-order exploratory factor analysis (DSEFA), which uses PSEM.

Input for Bi-Factor Confirmatory Factor Analysis Using 1 General and 4 Specific Factors for H&S19

```
MODEL:      ! CFA
            spatial BY visual-flags*;
            verbal BY general-wordm*;
            speed BY addition-straight*;
            memory BY wordr-figurew*;
            g BY visual-figurew*;
            spatial-g@1;
            ! uncorrelated factors:
            g WITH spatial-memory @0;
            spatial-memory WITH spatial-memory@0;
```

- CFA model fit with 1 + 4 factors, both schools (N=301) using MLR:
  Chi-square (133) = 242, p = 0.0000

Slide 32 shows the input for the bi-factor CFA model for the H&S19 data that we saw the model diagram for earlier. The 5 factors are specified to be uncorrelated in line with the diagram. The model fit is shown at the bottom.

Exploratory Bi-Factor Analysis: BI-GEOMIN Rotation

Bi-Factor EFA

```
ANALYSIS:
        TYPE = EFA 5 5;
        ROTATION = BI-GEOMIN;
```

Bi-Factor ESEM

```
ANALYSIS:
        ROTATION = BI-GEOMIN;
MODEL:
        fg f1-f4 by y1-y19(*1);
```

- BI-GEOMIN uses the Jennrich & Bentler (2011, 2012) method
- Overview of bifactor analysis in Reise, Mansolf, Haviland (2023)
  - Waller (2018): Direct Schmid-Leiman
- New approach in Mplus 9.1: Direct effects second-order factor analysis (DSEFA) using PSEM priors (Asparouhov & Muthén, 2026)

Slide 33 shows input for exploratory bi-factor analysis using TYPE = EFA as well as using the equivalent ESEM specification. For ESEM, the general factor is called fg. Alternatively, this could be written as f1-f5 BY y1-y19(*1).

Both approaches use the BI-GEOMIN rotation, which is based on the Jennrich-Bentler Gradient Projection Algorithm (the Psychometrika references are given in the Reference section).

A very helpful overview of bi-factor analysis, both confirmatory and exploratory, is given in the chapter by Reise and others in the SEM handbook of 2003 (see reference section). That chapter also refers to a paper by Waller which inspired the new DSEFA approach in Mplus discussed in Asparouhov- Muthén (2026) and presented next. The DSEFA approach uses PSEM priors on direct effects from a second-order factor to the factor indicators to arrive at a new type of bi-factor analysis.

Exloratory Bi-Factor Analysis using Direct Effects Second-Order Factor Analysis (DSEFA)

- Extends SEFA by letting the second-order factor have direct effects on the factor indicators where the effects have ALF priors
- Same model fit as regular EFA with 1 more factor than first-order SEFA factors
- First-order factor loadings have GEOMIN priors as with SEFA
- Significant direct effects suggest the need for DSEFA
- General factor loadings are computed as a sum of indirect and direct effects of the second-order factor on the indicators
- Avoids failures of ROTATION = BI-GEOMIN which are possible in cases with few indicators per factor and no indicators loading mostly on the general factor
- Pre-standardization of indicators important with widely varying variances so that the priors are used optimally
- Available also for categorical indicators using WLSMV estimation

Slide 34 shows the features of DSEFA.

It extends SEFA by letting the second-order factor have direct effects on the factor indicators where the effects have ALF priors.

It has the same model fit as regular EFA with 1 more factor than the number of first-order SEFA factors.

First-order factor loadings have GEOMIN priors as with SEFA.

Significant direct effects suggest the need for DSEFA.

General factor loadings are computed as a sum of indirect and direct effects of the second-order factor on the indicators.

DSEFA avoids failures of ROTATION = BI-GEOMIN which are possible in cases with few indicators per factor and no indicators loading mostly on the general factor.

Pre-standardization is important with widely varying indicator variances.

DSEFA is also available for categorical indicators using the WLSMV estimator.

Input for Exploratory Bi-Factor Analysis Using DSEFA: Direct Effects Second-Order Factor Analysis

```
ANALYSIS:            ESTIMATOR = MLR;
                     ITERATIONS = 10000;
                     CONVERGENCE = 0.000001;
                     !STARTS = 50;

MODEL:               ! first 3 lines below same as SEFA:
                     f1-f4 BY visual-figurew*(a1-a76); ! 4*19=76
                     f1-f4 (v1-v4);
                     f0 BY f1-f4* (l1-l4);
                     ! direct effects from the second-order
                     ! factor to the factor indicators:
                     f0 BY visual-figurew*0 (d1-d19);
                     f0@1;

MODEL CONSTRAINT:    v1 = 1 - l1*l1;
                     v2 = 1 - l2*l2;
                     v3 = 1 - l3*l3;
                     v4 = 1 - l4*l4;

MODEL PRIORS:        a1-a76 ~ GEOMIN(4,0.1);
                     ! impose alignment function (ALF) priors:
                     d1-d19 ~ ALF(0,1);
```

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Slide 35 shows the input for DSEFA. The ANALYSIS command includes settings for iterations, convergence, and starts that may be needed in the manual approach but not in the simplified, automatic approach given on the next slide.

In the MODEL command, the first 3 lines are the same as for SEFA. The specification of the second-order factor is bolded. After this, the direct effects from the second-order factor to the indicators are specified using the BY option (also bolded). Labels that are given for these effects (loadings) and used in the MODEL PRIORS command for the ALF priors.

The ALF priors have mean 0 and variance 1. Here, variance actually is 1/weight where the weight refers to the amount of penalty of the penalized maximum-likelihood estimation. The smaller the prior variance, the further the direct effects are pushed toward their means of zero. For a very small prior variance, it is as if there are no direct effects and the model is a SEFA. A variance of 1 is often a suitable choice. If necessary, it should be varied until the fit of the DSEFA is the same as for EFA with the same number of factors (here 5). For factor indicators that have widely varying variances, standardization of the indicators using the DEFINE command is recommended for the ALF priors to work well.

Simplified DSEFA Input

- Automated setup of model and priors
  - Special rotation
  - ESEM specification

```
ANALYSIS:   ESTIMATOR = MLR;
            ROTATION = DSEFA;
            ! DSEFA settings shown at the
            ! end of the Theory section

MODEL:      fg f1-f4 by visual-figurew(*1);
            ! covariates can be added
```

- Pre-standardization of factor indicators recommended
- EFA-generated starting values
- DSEFA-specific output
- Current limitations
  - Single group, at least 3 first-order factors, single EFA block

Slide 36 shows the simplified, automated DSEFA input. DSEFA is initiated by a special rotation called DSEFA.

The MODEL command uses an ESEM style specification of the factors where the first factor is the general one.

Pre-standardization of the factor indicators is recommended.

EFA-generated starting values are provided so the special ANALYSIS settings mentioned on the previous slide are typically not needed.

The DSEFA output includes the total effects of the second-order factor on the indicators, computed as the sum of the indirect effects and the direct effects. See also the EFA Theory section on the SEFA - DSEFA transition.

Currently DSEFA is limited to 1 group. At least 3 first-order indicators are needed. And, there cannot be more than one EFA block.

Results for Exploratory Bi-Factor Results Using DSEFA

STANDARDIZED SOLUTION

BI-FACTOR MODEL

	General	Spatial	Verbal	Speed	Memory
VISUAL	**0.432***	**0.528***	**0.185***	0.013	0.062
CUBES	**0.267***	**0.397***	0.062	-0.119	-0.023
PAPER	**0.222***	**0.420***	**0.151***	0.012	-0.034
FLAGS	**0.284***	**0.585***	0.005	0.027	0.131
GENERAL	**0.345***	0.031	**0.767***	0.082	-0.027
PARAGRAP	**0.428***	0.016	**0.695***	0.025	0.087
SENTENCE	**0.507***	-0.116	**0.728***	-0.018	-0.068
WORDC	**0.476***	0.031	**0.574***	0.014	0.039
WORDM	**0.374***	0.110	**0.759***	0.014	0.073
ADDITION	0.121	-0.024	0.087	**0.771***	0.130
CODE	**0.508***	0.033	0.105	**0.453***	0.138
COUNTING	**0.283***	0.180	0.012	**0.584***	-0.024
STRAIGHT	**0.586***	0.262	-0.062	**0.384***	-0.081
WORDR	0.337	-0.060	0.018	-0.045	**0.580***
NUMBERR	0.118	0.157	-0.041	0.023	**0.577***
FIGURER	0.416	0.259	0.039	-0.003	**0.403***
OBJECT	**0.222***	-0.027	-0.009	**0.310***	**0.509***
NUMBERF	0.222	0.167	0.064	0.182	**0.404***
FIGUREW	**0.354***	0.044	0.113	0.021	0.261

- All factors are uncorrelated
- The 4 specific factors are well recovered
- ROTATION = DSEFA has the same fit as regular 5-factor EFA and ROTATION = BI-GEOMIN. It is just a different rotation

Slide 37 shows the DSEFA estimates. This output is presented under the headings STANDARDIZED SOLUTION, BI-FACTOR MODEL. All the factors are uncorrelated.

The general factor has mostly significant loadings. The 4 specific factors Spatial, Verbal, Speed, and Memory all have large significant loadings in expected places.

ROTATION = DSEFA has the same fit as regular 5-factor EFA and ROTATION = BI-GEOMIN. It is just a different rotation.

Comparison with Exploratory Bi-Factor Results Using ROTATION = BI-GEOMIN (TYPE = EFA or ESEM)

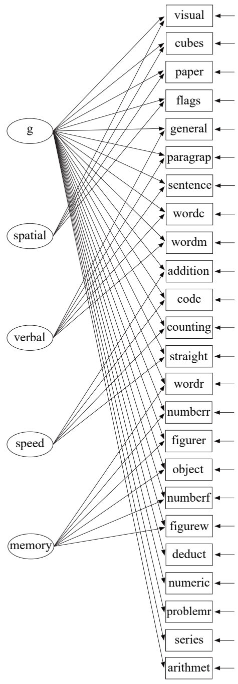
ROTATED LOADINGS (* significant at 5% level)

	General	Verbal	Speed?	Memory	?
VISUAL	0.619*	0.055	-0.339*	-0.011	-0.016
CUBES	0.340*	-0.021	-0.371*	-0.044	0.019
PAPER	0.407*	0.046	-0.249	-0.123	-0.081
FLAGS	0.544*	-0.143	-0.332*	0.021	-0.114
GENERAL	0.420*	**0.748***	0.057	-0.098	-0.053
PARAGRAP	0.436*	**0.687***	0.001	0.064	0.024
SENTENCE	0.367*	**0.763***	-0.002	-0.013	0.191
WORDC	0.438*	**0.569***	-0.046	0.049	0.103
WORDM	0.464*	**0.730***	-0.034	-0.005	-0.095
ADDITION	0.476*	-0.003	0.634*	-0.029	-0.045
CODE	0.595*	0.054	0.259*	0.131	0.222
COUNTING	0.564*	-0.105	0.309*	-0.130	0.080
STRAIGHT	0.670*	-0.168	0.011	-0.063	0.335
WORDR	0.247*	0.057	-0.002	**0.629***	0.053
NUMBERR	0.263*	-0.073	-0.010	**0.493***	-0.209
FIGURER	0.476*	-0.011	-0.178*	**0.388***	0.016
OBJECT	0.355*	-0.029	0.287*	**0.450***	-0.042
NUMBERF	0.407*	0.016	0.082	**0.308***	-0.143
FIGUREW	0.321*	0.114	-0.038	**0.285***	0.081

- Spatial and speed factors not recovered: Failure of ROTATION=BI-GEOMIN
- Same fit as regular 5-factor EFA: Chi-2(86) = 113, p = 0.028

Slide 38 uses the older approach of ROTATION = BI-GEOMIN (Jennrich & Bentler, 2011, 2012). Although the model fit for ROTATION = BI-GEOMIN is the same as for ROTATION = DSEFA on the previous slide, the solution using ROTATION = BI-GEOMIN does not give the same quality of results. With this rotation, only the Verbal and Memory factors are well recovered, whereas the Spatial factor is not recovered and the Speed factor is poorly recovered.

Holzinger-Swineford 24-Variable Bi-Factor Model



Slide 39 shows a bi-factor model diagram for the H&S data extended by 5 tests to give 24 factor indicators. The 5 test are at the bottom of the diagram. The diagram shows them loading only on the general factor.

Input for H&S 24-Variable Bifactor Analysis Using CFA, TYPE=EFA, and ESEM

```
MODEL:      ! CFA
            spatial BY visual-flags*;
            verbal BY general-wordm*;
            speed BY addition-straight*;
            memory BY wordr-figurew*;
            g BY visual-arithmet*;
            spatial-g@1;
            ! uncorrelated factors because of the general factor:
            g WITH spatial-memory @0;
            spatial-memory WITH spatial-memory@0;

ANALYSIS:   ! EFA
            ESTIMATOR = ML;
            TYPE = EFA 5 5;
            ROTATION = BI-GEOMIN;

ANALYSIS:   ! ESEM
            ESTIMATOR = ML;
            ROTATION = BI-GEOMIN;
MODEL:      fg f1-f4 BY visual-arithmet(*1);
```


Slide 40 shows 3 inputs for these data: CFA, EFA, and ESEM. The EFA and ESEM results using ROTATION = BI-GEOMIN will be compared to DSEFA.

Input for H&S 24-Variable Bifactor Analysis Using DSEFA

```
ANALYSIS:   ESTIMATOR = MLR;
            ROTATION = DSEFA;
            ! DSEFA settings shown at the
            ! end of the Theory section

MODEL:      fg f1-f4 by visual-arithmet(*1);
            ! covariates can be added
```


Slide 41 shows the DSEFA input for the 24-variable analysis. DSEFA is defined by ROTATION = DSEFA. The DSEFA option can have several settings related to the priors. These are described in the EFA Theory section. The MODEL statement uses an ESEM type specification.

Model Fit for H&S 24-Variable Bifactor Models Grant-White and Pasteur Schools, N = 301, MLR

SUMMARY OF MODEL FIT INFORMATION

Model	Number of Parameters	Chi-Square	Degrees of Freedom	P-Value
4-factor	138	283.322	186	0.0000
5-factor	158	238.592	166	0.0002

Models Compared	Chi-Square	Degrees of Freedom	P-Value
4-factor against 5-factor	38.307	20	0.0081

Slide 42 shows EFA model test results for the 24-variable analysis. In contrast to the 19-variable case, the fit is not good for 4 factors and the 5-factor solution is chosen here. For the bi-factor analyses, a 5-factor solution corresponds to one general and 4 specific factors.

The fit of the 5-factor model can be improved by searching for significant residual covariances. With PSEM, modification indices are not available to guide this search. A simple approach is to instead use an ESEM EFA analysis with 5 factors requesting modindices since the fit of this model is the same as for the DSEFA. This pointed to a significant residual covariance between figurew and addition. When this was freed in the DSEFA analysis, a better model fit was obtained: chi-square(165) = 202 (p = 0.025).

Findings from Second-Order and Bi-factor Models using the 24-Variable H&S Data

- Competing models:
  - SEFA using 4 first-order factors with the same fit as regular 4-factor EFA
  - DSEFA using 4 first-order factors and a second-order factor with direct effects on the indicators with the same fit as regular 5-factor EFA
- DSEFA shows significant direct effects from the second-order factor to 3 of the 5 indicators added in the 24-variable version which suggests that SEFA is not sufficient but DSEFA needed (compare 4- vs 5-factor testing)
- In the 24-variable case, ROTATION = BI-FACTOR and ROTATION = DSEFA give similar and interpretable solutions, perhaps due to the 5 extra indicators loading mainly on the general factor. This agreement is in line with Asparouhov-Muthén (2026) simulations
- For more on the relationship between SEFA and DSEFA, see the EFA theory section slide SEFA-DSEFA transition

Slide 43 discusses results from second-order and bi-factor analysis for the 24-variable H&S data.

We consider two competing models:

- SEFA using 4 first-order factors with the same fit as regular 4-factor EFA

- DSEFA using 4 first-order factors and a second-order factor with direct effects on the indicators with the same fit as regular 5-factor EFA

DSEFA shows significant direct effects from the second-order factor to 3 of the 5 indicators added in the 24-variable version, which suggests that SEFA is not sufficient but DSEFA is needed (compare 4- vs 5-factor EFA testing).

In the 24-variable case, ROTATION = BI-FACTOR and ROTATION = DSEFA give similar and interpretable solutions, perhaps due to the 5 extra indicators loading mainly on the general factor. The agreement in this case is in line with Asparouhov-Muthén (2026) simulations.

Slide 135 in the EFA theory section says a bit about the interpretation of the different factor loading matrices for SEFA and DSEFA and how the models relate to each other.

DSEFA Solution for 24-Variable H&S

STANDARDIZED SOLUTION

BI-FACTOR MODEL

	General	Spatial	Verbal	Speed	Memory
VISUAL	0.415*	0.531*	0.182*	0.109	0.046
CUBES	0.274*	0.424*	0.044	-0.064	-0.047
PAPER	0.251*	0.370*	0.112	0.075	-0.069
FLAGS	0.367*	0.535*	-0.057	0.086	0.070
GENERAL	0.398*	-0.009	0.728*	0.040	-0.068
PARAGRAP	0.442*	0.026	0.690*	0.005	0.062
SENTENCE	0.360*	0.007	0.808*	0.012	-0.025
WORDC	0.427*	0.077	0.602*	0.022	0.039
WORDM	0.474*	0.068	0.695*	-0.040	0.012
ADDITION	0.424	-0.317	-0.052	0.609*	0.003
CODE	0.370*	0.040	0.206*	0.525*	0.184*
COUNTING	0.378*	0.039	-0.023	0.540*	-0.062
STRAIGHT	0.348*	0.307*	0.079	0.522*	0.006
WORDR	0.281*	-0.005	0.067	-0.025	0.596*
NUMBERR	0.233*	0.105	-0.103	0.006	0.516*
FIGURER	0.462*	0.248*	0.017	0.012	0.369*
OBJECT	0.254*	-0.054	-0.004	0.287*	0.500*
NUMBERF	0.285*	0.087	0.037	0.180*	0.380*
FIGUREW	0.408*	0.046	0.101	-0.017	0.237*
DEDUCT	0.521*	0.266*	0.155	-0.137*	0.102
NUMERIC	0.664*	0.111	0.048	0.140	-0.025
PROBLEMR	0.529*	0.233	0.320*	-0.057	0.018
SERIES	0.674*	0.278*	0.179*	-0.021	-0.001
ARITHMET	0.689*	-0.121	0.172	0.088	0.056

- The 5 extra tests load mainly on the general factor

Slide 44 shows the DSEFA solution for the 24-variable case. This output is presented under the headings STANDARDIZED SOLUTION, BI-FACTOR MODEL. All the factors are uncorrelated.

The 4 specific factors are all well recovered. The general factor has significant loadings for all variables. The 5 extra variables mainly load on the general factor with some cross loadings.

2.2 Hypothesis about the number of factors and key items

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors
  - Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM
- **Hypothesis about the number of factors and key items**:
  - **ESEM with Target rotation**
  - PSEM with ALF priors for cross loadings
- Comparing EFA methods
- Special models:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings

Slide 45 returns to the EFA Variations overview where we now turn to analyses where not only the number of factors is specified but also key items. This was referred to as the middle analysis stage on slide 4.

The first topic is ESEM with target rotation.

2.2.1 ESEM with Target rotation

ESEM with Target Rotation

- A stage in between of EFA and CFA: Rotation guided by substantive considerations, not using mechanical EFA rotation
- Same model fit as EFA, just a different rotation
- Choose rotation by giving target loading values (typically zero)
- Target values not fixed as in CFA — zero targets can come out big if misspecified
- Minimum requirement for identification is $m-1$ (m = # factors) zeros in each loading column which gives EFA which together with factor variances fixed at 1 results in the m^2 restrictions $m(m-1)+m$
- Mplus language using ESEM:
  - ```
    f1 BY y1-y10 y1~0 (*1);
    f2 BY y1-y10 y5~0 (*1);
    ```
- Tucker (1944), Browne (1972a, b). Asparouhov & Muthén (2024) suggests that PSEM does better than Target

Slide 46 discusses Target rotation. Target rotation represents an analysis stage that lies in between of EFA and CFA: The rotation is guided by substantive considerations, not using mechanical EFA rotation.

It has the same model fit as EFA, just using a different rotation. You choose a rotation by giving target loading values (typically zero).

Target values are not fixed as in CFA — zero targets can come out big if misspecified.

Minimum requirement for identification is $m-1$ (where $m = \#$factors) zeros in each loading column which gives EFA which together with factor variances fixed at 1 results in the $m(m-1)+m = m^2$ restrictions.

The Mplus language uses ESEM for one factor at a time using the label 1 to denote the same EFA block. A curl $\sim$ is used together with a value to denote a target value for a loading, typically zero.

At the bottom are two early references for this technique and also the 2024 PSEM paper by us. As we will see, there are now better approaches to target rotation.
    ```

ESEM Target Input and Results for the H&S example

```
ANALYSIS:  ESTIMATOR = MLR;
           ROTATION = TARGET;

MODEL:     spatial BY visual-flags general-figurew~0 (*1);
           verbal BY visual-flags~0 general-wordm addition-figurew~0 (*1);
           speed BY visual-wordm~0 addition-straight wordr-figurew~0 (*1);
           memory BY visual-straight~0 wordr-figurew (*1);
           spatial-memory@1;
```

Grant-White has 9 significant cross-loadings, Pasteur has 12

Slide 47 shows the target input for the H&S example with 19 variables. The ANALYSIS uses ROTATION = TARGET. The MODEL command uses the loading pattern of slide 9 to give zero target values to all cross loadings.

This analysis results in 9 significant cross loadings for the Grant- White school and 12 for Pasteur. This is more than the 5 and 8 that we saw with regular EFA.

ESEM Bi-Factor Target Input for the H&S Example

```
ANALYSIS: ESTIMATOR = MLR;
          ROTATION = TARGET(ORTHOGONAL);

MODEL:    spatial BY visual-flags general-sentence~0 wordc-figurew(*1);
          verbal BY visual-paper~0 flags-figurew(*1);
          speed BY visual-straight wordr-figurer~0 object-figurew(*1);
          memory BY visual-wordm addition-counting~0 straight-figurew(*1);
          g BY visual-figurew(*1);
```

- This will be referred to as Model M5 in a later table

Slide 48 shows that Mplus also offers target rotation with bi-factor modeling. In the ANALYSIS command, this uses ROTATION = TARGET(ORTHOGONAL) so that all factors are uncorrelated. The MODEL command adds the general factor called g to the ESEM specification. On slides 56 and 57 this is the specification referred to as M5.

2.2.2 PSEM with ALF priors for cross loadings

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors
  - Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM
- Hypothesis about the number of factors and key items:
  - ESEM with Target rotation
  - **PSEM with ALF priors for cross loadings**
- Comparing EFA methods
- Special models:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings

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Slide 49 shows the EFA variations overview again and we see that we will now turn to using PSEM with ALF priors for cross loadings.

PSEM with ALF Priors for Cross-Loadings for the 4-Factor H&S19 Example

- Specify the CFA factors
- Label the cross loadings
- Give ALF priors to the cross loadings

```
ANALYSIS:        ESTIMATOR = MLR;

MODEL:           spatial BY visual-flags*1
                        general-figurew*0 (a1-a15);
                 verbal BY visual-flags*0 (b1-b4)
                        general-wordm*1
                        addition-figurew*0 (c1-c10);
                 speed BY visual-wordm*0 (d1-d9)
                        addition-straight*1
                        wordr-figurew*0 (e1-e6);
                 memory BY visual-straight*0 (f1-f13)
                        wordr-figurew*1;
                 spatial-memory@1;

MODEL PRIORS:    a1-f13~ALF(0,1.0);
```


Slide 50 shows the input for PSEM with ALF priors for cross loadings. There are 3 parts to this: Specify the CFA factor loadings, label the cross loadings, and give ALF priors for the cross loadings in MODEL PRIORS.

The cross loadings are the same loadings that were given zero targets in the Target rotation approach. With ALF priors, the targets need not be exactly zero but approximately so by varying the variance of the ALF prior. Target rotation would correspond to using a very small ALF prior variance so that the values are essentially held at the prior mean of zero. The ALF prior variances can also be different for different cross loadings.

The intention of the PSEM ALF cross-loadings analysis is that its log likelihood value should match that of the 4-factor EFA so that PSEM ALF essentially provides a different rotation. To accomplish this, a variance of 1 is often a suitable choice for the ALF prior variance but this should be checked.

For factor indicators that have widely varying variances, standardization of the indicators using the DEFINE command is recommended for the ALF priors to work well.

Comparing EFA, Target, and PSEM ALF for the 4-Factor H&S19 Example

- Log likelihood is the same for the 3 approaches when those runs standardize the variables, but different rotations are used
- Number of significant cross loadings for Grant-White/Pasteur:
  - EFA (GEOMIN): 5/8
  - Target: 9/12
  - PSEM ALF (v=1): 5/7

Slide 51 gives a comparison of EFA, Target rotation, and PSEM ALF cross loadings for the H&S example with 19 variables. The model fit is the same - the log likelihoods agree when the variables are standardized before analysis.

But, the 3 methods use different rotations producing different numbers of significant cross loadings for Grant-White/Pasteur: 5/8, 9/12, 5/7. The PSEM ALF cross loading approach appears to give the simplest solution.

Target Simulations in Asparouhov-Muthén (2024)

- Section 4.4 comparison of PSEM and ESEM alternatives (N = 500)
- This web talk adds PSEM Geomin and ESEM Geomin results to Table 4 (BSEM dropped)

$$\Lambda = \begin{array}{c} \begin{matrix} \text{F1} & \text{F2} \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0.3 & 1 \end{bmatrix} \end{array}, \qquad \Psi = \begin{bmatrix} 1 & \\ 0.5 & 1 \end{bmatrix} \tag{1}$$

- Critique of target rotation for the case of incorrect target

Slide 52 describes a simulation study in our 2024 PSEM paper. The simulations support that using PSEM ALF priors for cross loadings is preferred over Target rotation.

The model used for the simulations is shown in equation (1). The simple loading pattern is broken by one cross loading that appears for the last variable and the first factor. The premise is that the Target rotation incorrectly specified a zero target for this loading.

Methods Compared

- PSEM ALF: Cross-loadings given ALF(0,1) priors as 3 slides earlier
- PSEM LASSO: Cross-loadings given LASSO(0,1) priors
- PSEM Normal: Cross-loadings given N(0,1) priors
- PSEM Geomin: All loadings given GEOMIN(2,1,0.0001) priors
- ESEM Geomin (similar to PSEM Geomin):
  - ANALYSIS: ROTATION = GEOMIN;
  - MODEL: f1-f2 BY y1-y6 (*1);
- ESEM Target:
  - ANALYSIS: ROTATION = TARGET;
    MODEL: f1 BY y1-y3 y4-y6~ 0 (*1);
    f2 BY y1-y3~0 y4-y6 (*1);

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Slide 53 lists the different approaches that are studied. The preferred PSEM ALF approach is at the top. The target rotation approach is listed last showing the incorrect zero target for the y6 loading of the f1 factor.

Table 4 Extended: Abs Bias(Coverage)

Par.	True Value	PSEM ALF	PSEM LASSO	PSEM Normal	PSEM Geomin	ESEM Geomin	ESEM Target
λ_{11}	1	.00(.93)	.00(.93)	.01(.93)	.00(.94)	.00(.93)	.00(.93)
λ_{12}	0	.00(1.0)	.00(1.0)	.01(.94)	.00(1.0)	.00(.99)	.01(.95)
λ_{61}	.3	.02(.99)	.03(.92)	.09(.27)	.02(.97)	.02(.99)	.09(.27)
λ_{62}	1	.00(.95)	.00(.95)	.02(.91)	.00(.95)	.00(.95)	.02(.91)
ψ_{12}	.25	.00(.96)	.02(.94)	.06(.71)	.00(.95)	.01(.96)	.07(.63)

- Conclusion:
  - PSEM ALF, PSEM GEOMIN, and ESEM GEOMIN perform best
  - PSEM Normal and ESEM Target worst (see λ_{61})
- Target problems and iterative target discussed in Moore, Reise, Depaoli, & Haviland (2015)

Slide 54 shows the table of results. This is Table 4 of Section 4.4 of the 2024 PSEM paper. The critical loading λ_{61} is particularly sensitive to the method used. The target approach (ESEM Target) gets a large bias and poor coverage for this loading.

The overall conclusion is that PSEM ALF, PSEM GEOMIN, and ESEM GEOMIN perform best while PSEM Normal and ESEM Target perform the worst.

Problems with target rotation has been studied in the literature before, with suggestions to use an iterative target approach where target loadings that obtain large estimates are dropped as targets in a second target analysis. An overview of this topic is given in the Moore et al. (2015) paper given in the reference section.

2.3 Comparing EFA methods

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors
  - Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM
- Hypothesis about the number of factors and key items:
  - ESEM with Target rotation
  - PSEM with ALF priors for cross loadings
- **Comparing EFA methods**
- Special models:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings

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Slide 55 returns to the EFA Variations overview. Here we turn to comparisons of exploratory second-order and bi-factor analyses based on simulation results. Target methods are also included.

Comparing EFA Methods: Data Generated from a 3-Factor SEFA Methods Sorted by MSE Performance

- Monte Carlo simulation study in Asparouhov & Muthén (2026)

Method	Rotation	Model	Framework	Number only?	Conv	Avg MSE	Avg Coverage
M1	Geomin	SEFA	PSEM	Yes	100%	0.0033	92.9%
M2	Target-alf	SEFA	PSEM	No	100%	0.0036	94.7%
M3	Geomin/alf	DSEFA	PSEM	Yes	100%	0.0039	94.6%
M4	Target-alf	Bifactor	PSEM	No	85%	0.0131	91.0%
M5	Target	Bifactor	ESEM	No	60%	0.0134	86.7%
M6	Geomin	EFA	PSEM	Yes	88%	0.0164	84.2%
M7	Bi-geomin	Bifactor	PSEM	No	81%	0.0191	86.0%
M5	Target	Bifactor	ESEM	No	100%	0.0222	34.1%
M8	Bi-geomin	Bifactor	ESEM	Yes	60%	0.0256	52.1%

- Number only? refers to specifying only the number of factors, not also key loadings
- M1 is the SEFA model discussed earlier
- M2 is PSEM ALF discussed earlier but adding a second-order factor
- M3 is the DSEFA model discussed earlier

Slides 56 and 57 show simulation results from the Asparouhov & Muthén 2026 paper: A unification of second-order and bi-factor EFA. The methods are listed in the order of performance from best to worst. The performance of the methods is judged by convergence percentage, average mean square error, and average coverage.

In the table on slide 56, the data are generated from a 3-factor SEFA and it is natural that this method, which is method M1, performs the best. Method M2 performs second best but needs not only specification of the number of factors but also locations of the targets.

Comparing EFA Methods: Data Generated from a 4-Factor Bi-Factor Model Methods Sorted by MSE Performance

- Monte Carlo simulation study in Asparouhov & Muthén (2026)

Method	Rotation	Model	Framework	Number only?	Conv	Avg MSE	Avg Coverage
M3	Geomin/alf	DSEFA	PSEM	Yes	100%	0.0069	93.8%
M4	Target-alf	Bifactor	PSEM	No	99%	0.0152	97.1%
M1	Geomin	SEFA	PSEM	Yes	98%	0.0167	93.4%
M5	Target	Bifactor	ESEM	No	97%	0.0255	87.2%
M8	Bi-geomin	Bifactor	ESEM	Yes	97%	0.0270	93.5%

- Number only? refers to specifying only the number of factors, not also key loadings
- M4 is the bi-factor PSEM ALF shown on the next slide
- M5 is the bi-factor target ESEM shown on slide 47

- M1 and M3 are among the top 3 in both tables and require only the number of factors
- M2 and M4 are among the top when key loadings are also specified

In the table of slide 57, the data are generated from a 4-factor bi-factor model so that method M3 performs best.

Drawing on the results from both tables, we conclude that the SEFA and DSEFA methods M1 and M3 are doing well - and importantly - require only the specification of the number of factors. Methods M2 and M4 do well when the location of key loadings are also specified. It is notable that M8 performs the worst in this simulation. It uses ROTATION = BI-GEOMIN which has to date been the most commonly used.

Input for Bi-Factor PSEM using ALF Cross-Loading Priors (M4)

```
ANALYSIS:       ESTIMATOR = MLR;
                ITERATIONS = 10000;
                CONVERGENCE = 0.000001;
                STARTS = 50;

MODEL:          ! 4 specific factors
                spatial BY
                  visual-flags*1
                  general-arithmet*0 (a1-a20);

                verbal BY
                  visual-flags*0 (b1-b4)
                  general-wordm*1
                  addition-arithmet*0 (c1-c15);

                speed BY
                  visual-wordm*0 (d1-d9)
                  addition-straight*1
                  wordr-arithmet*0 (e1-e11);

                memory BY
                  visual-straight*0 (f1-f13)
                  wordr-figurew*1
                  deduct-arithmet*0 (g1-g5);

                ! general factor:
                gen BY visual-arithmet*;

                spatial-gen WITH spatial-gen@0;
                spatial-gen@1;

MODEL PRIORS:   a1-g5 ~ ALF(0,1);

OUTPUT:         STANDARDIZED;
```


For completeness, slide 58 shows the input for bi-factor PSEM with ALF cross loadings priors. This is method M4 in the previous tables. The input refers to the 24-variable H&S data.

2.4 Special models

EFA Variations

- Hypothesis about the number of factors:
  - ANALYSIS: TYPE = EFA
  - ESEM (*1)
  - PSEM with GEOMIN priors
  - Second-order exploratory factor analysis (SEFA) using PSEM with GEOMIN priors
  - Bi-factor analysis using ROTATION = BI-GEOMIN and direct second-order exploratory factor analysis (DSEFA) using PSEM
- Hypothesis about the number of factors and key items:
  - ESEM with Target rotation
  - PSEM with ALF priors for cross loadings
- Comparing EFA methods
- **Special models**:
  - ESEM with PSEM priors for residual covariances
  - PSEM finding a small number of cross-loadings

Slide 59 again shows the overview of EFA Variations. We now turn to the last 2 items on the list under Special models: ESEM where all residual covariances are included and PSEM searching for a small number of cross loadings.

ESEM with PSEM Priors for Residual Covariances

- PSEM makes it possible to allow **all** residual covariances
- Significant residual covariances can then be freed

```
MODEL:          f1-f2 BY y1-y10 (*1);
                y1-y10 WITH y1-y10 (c1-c45);

MODEL PRIORS:   c1-c45~ALF(0,1);
```

- See Section 4.5 of Asparouhov & Muthén (2024)

Slide 60 shows that it is possible to include residual covariances in an EFA by using PSEM. In the previous British Household Panel example we used ESEM to include some residual covariances. With PSEM, however, we can include all of them. This makes it possible to see which of them are significant and should be freed.

As the input shows, ALF priors can be applied to all the residual covariances. These priors push the residual covariances toward the ALF prior mean of zero. If the dataset clearly requires a residual covariance, it will overwhelm the prior and show a substantial non-zero residual covariance estimate.

The technical background for this is discussed in section 4.5 of the 2024 Asparouhov-Muthén article on PSEM.

It is not recommended to use this approach in conjunction with more complex models such as SEFA and DSEFA. The alternative approach of using modindices with regular EFA was presented on slide 42.

PSEM Used for Finding the Smallest Number of Essential Cross-Loadings: Transition from EFA to CFA

- Strategy:
  - Start from an EFA and check the number of significant cross loadings
  - Continue with a PSEM CFA that adds cross-loadings with ALF priors
    - Specify ALF priors with a variance that makes the loglikelihood match that of EFA (typically variance = 1.0)
    - Specify ALF priors for cross-loadings with decreasing variance until the loglikelihood decreases significantly
  - Do a CFA which frees the cross-loadings that are significant in the previous run and see if BIC has improved compared to EFA

Slide 61 shows a different use of PSEM. It can answer the question of which is the smallest set of essential cross loadings when transitioning from an EFA to a CFA. BIC is used as the basis for the decision.

The strategy is to start from an EFA where you check the number of significant cross loadings.

You then continue with a PSEM CFA that adds cross-loadings with ALF priors.

- First you specify ALF priors with a variance that makes the loglikelihood match that of EFA (typically variance = 1.0).

- Then you specify these ALF priors for cross-loadings with decreasing variance until the loglikelihood decreases substantially.

Finally, you do a CFA where you free the cross-loadings that are significant in the previous run and see if BIC has improved compared to EFA.

Example: NELS Data (N = 5198)

- National Education Longitudinal Study, eighth graders in urban Catholic and Public schools (Muthén et al., 1997)

Variable	Reading	Math	Science	HCG
Y1	X	0	0	0
Y2	X	0	0	0
Y3	X	0	0	0
Y4	X	0	0	0
Y5	X	0	0	0
Y6	0	X	0	0
Y7	0	X	0	0
Y8	0	X	0	0
Y9	0	X	0	0
Y10	0	0	X	0
Y11	0	0	X	0
Y12	0	0	X	0
Y13	0	0	X	0
Y14	0	0	0	X
Y15	0	0	0	X
Y16	0	0	0	X

- Reading: literature, science, poetry, biography, history. Math: algebra, artihmetic, geometry, probability. Science: earth, chemistry, life, methods. HCG: history, geography, citizenship

Slide 62 shows the expected loading pattern for a measurement instrument used in the National Education Longitudinal Study of eighth graders in the US. The sample size is 5,198. Four factors are expected. The 16 factor indicators are described at the bottom of the table.

PSEM Input for NELS

- Specify the CFA factors
- Label the cross loadings
- Give ALF priors to the cross loadings

```
MODEL:
        f1 BY y1-y5*1
          y6-y16*0 (a6-a16);
        f2 BY y6-y9*1
          y1-y5*0 (b1-b5)
          y10-y16*0 (c10-c16);
        f3 BY y10-y13*1
          y1-y9*0 (d1-d9)
          y14-y16*0 (e14-e16);
        f4 BY y14-y16*1
          y1-y13*0 (f1-f13);
        f1-f4@1;
MODEL
PRIORS:
        a6-f13~ALF(0,1);
```


Slide 63 shows the input for the PSEM part of the exploration of essential cross loadings.
You specify the CFA factor pattern, label the cross loadings, and give ALF priors to those cross loadings.

NELS 4-Factor Solutions using EFA and PSEM

Model	# par's	LL	BIC	χ^2	Df	X-loads
1. EFA	90	-124527	249823	157	62	13
2. CFA	54	-124651	249764	394	98	0
3. CFA + x-loads PSEM ALF v=1.0	90	-124527	249823	156	62	11
4. CFA + x-loads PSEM ALF v=0.1	90	-124538	249847	187	62	6
5. CFA + 13 free x-loads of M1	67	-124545	249664	189	85	13
6. CFA + 11 free x-loads of M3	65	-124556	249668	211	87	11
7. CFA + 6 free x-loads of M4	60	-124566	**249645**	229	92	6

Slide 64 shows a series of models using EFA, CFA, and CFA with PSEM. Of particular importance are the loglikelihood (LL) and BIC values.

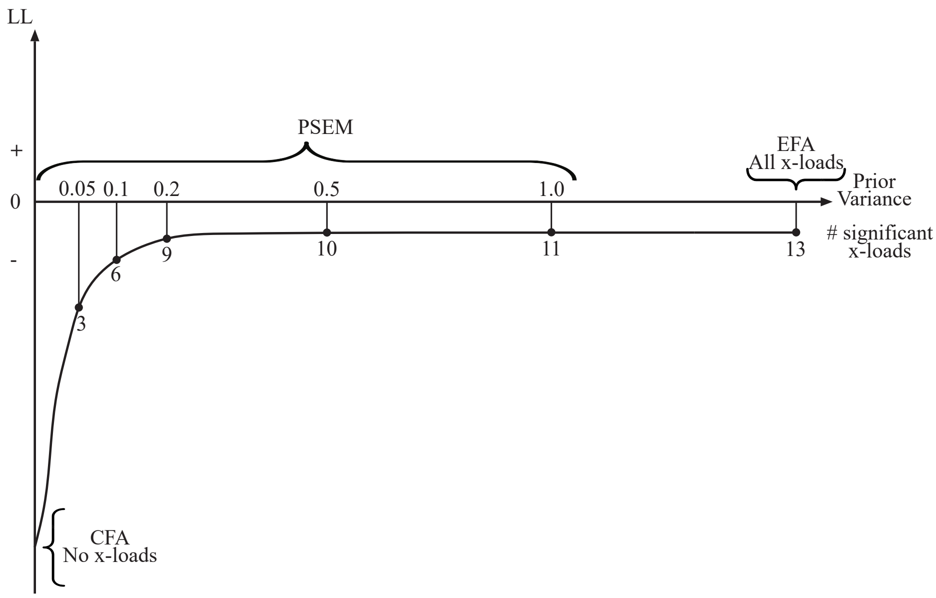
Model 1 is a 4-factor EFA which has 13 significant cross loadings.

Model 2 is a regular CFA. The CFA model has better (lower) BIC than the EFA.

Model 3 is a CFA adding all cross-loadings using ALF priors. The variance 1 results in the same model fit as the Model 1 EFA as evidenced by the log likelihood (LL). This results in 11 significant cross loadings, representing a modest reduction from the 13 found with EFA. Having found this model fit equivalence with EFA, smaller variances for the ALF priors can be explored.

Model 4 uses ALF prior variance of 0.1. As expected, this worsens the log likelihood. To gauge the magnitude of the drop, it is useful to compare it to the distance between the EFA and CFA log likelihoods. In those terms, the drop is only 8.9%. For this model, the number of significant cross loadings is only 6. Models with small log likelihood drops and a number of significant cross loadings smaller than that of EFA provide a basis for a follow-up CFA that may have a competitive BIC as illustrated in the bottom part of the table.

The bottom part of the table adds significant cross loadings to the regular CFA Model 2 based on models 1, 3, and 4. These 3 models give better (lower) BIC values than any of the models in the top part. Of particular importance is that Model 7 with only 6 cross loadings gives the best BIC and is therefore the model of choice.



Slide 65 shows a log likelihood curve for the different analyses on the previous slide. At top right is the EFA model which has all cross loadings of which 13 are significant. It can be seen as a PSEM model where the ALF prior variances are very large so that the priors essentially have no effect.

Bottom left is the CFA model which has no cross loadings. In between, there is a series of PSEM models with varying ALF prior variance and corresponding number of significant cross loadings. The plot shows the chosen model which has prior variance 0.1 with 6 significant cross loadings, representing the 8.9% drop in the log likelihood, and receiving the best BIC.

3 EFA in an SEM setting

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- **EFA in an SEM setting**
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- EFA theory

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Slide 66 returns to the overall Outline of the presentation, having covered EFA Variations. We now go beyond the focus on EFA of a measurement instrument in isolation and turn to the topic of EFA in a structural equation modeling setting. We will cover the case of MIMIC modeling, that is, adding covariates as well as discussing EFA in general SEM settings.

ESEM and PSEM Sources

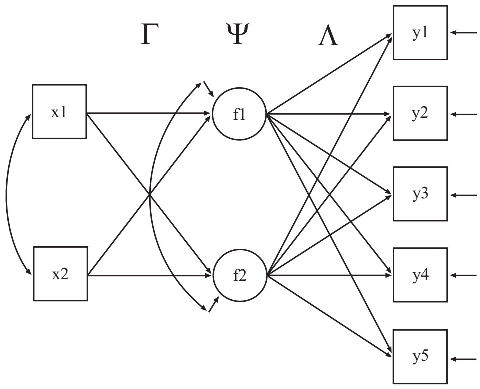
- ESEM:
  - Asparouhov & Muthen (2009). Exploratory structural equation modeling. *Structural Equation Modeling: A Multidisciplinary Journal*
  - Application papers at: `http://www.statmodel.com/ESEM.shtml`
- PSEM:
  - Asparouhov & Muthén (2024, 2025, 2026)
  - Muthén (2005)
  - Special Mplus Topics: `https://www.statmodel.com/psem.shtml`

Slide 67 lists some references for what we will talk about. ESEM will be a key approach for EFA in an SEM setting and this idea was presented in the 2009 Asparouhov-Muthén paper. As you see here, ESEM has its own Special Mplus Topics web page where you find several application papers.

The PSEM approach is also essential for EFA in an SEM setting. PSEM developments have appeared in papers and talks which are all listed on our Special Mplus Topics page for PSEM. This includes PSEM theory in our papers from 2024, 2025, and 2026 listed in the references.

3.1 ESEM MIMIC

Example: ESEM MIMIC



- Step 1: Unrotated loadings model, estimating Λ with $\Psi = I$ (standardized, uncorrelated factors), and the Γ regressions of factors on covariates
- Step 2: EFA rotation of the y measurement part gives new Λ, Ψ, and a corresponding transformation is applied to the estimated Γ from Step 1

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Slide 68 shows an example with an EFA measurement model that has 2 factors and 6 indicators, to which we add 2 covariates x_1 and x_2. The loading matrix is called Λ, the factor covariance matrix is called Ψ and the regression matrix is called Γ. Ψ is now actually a residual covariance matrix for the factors. This is called a MIMIC model where MIMIC stands for multiple indicators, multiple causes.

Typically, MIMIC modeling is done by using a CFA measurement model but here the ESEM approach makes it possible to use an EFA measurement model. ESEM uses two steps:

In Step 1, an unrotated loadings model estimates Λ with $\Psi = I$ (so standardized, uncorrelated factors), and the Γ regressions of factors on covariates.

In Step 2, an EFA rotation is made of the y measurement part and gives new Λ and Ψ, after which a transformation is applied to the estimated Γ from Step 1.

ESEM MIMIC Inputs

ESEM MIMIC

```
MODEL:
        f1-f5 BY y1-y15 (*1);
        f1-f5 ON age;
```

ESEM MIMIC with rescovs

```
MODEL
        f1-f5 BY y1-y15 (*1);
        y1 y5 y9 y12 WITH y1 y5 y9 y12;
        f1-f5 ON age;
```

Bi-Factor ESEM MIMIC

```
ANALYSIS:
        ROTATION = BI-GEOMIN;
MODEL:
        fg f1-f4 BY y1-y15(*1);
        fg-f4 ON age;
```


Slide 69 shows the generality of the ESEM MIMIC approach in three variations of the input: the EFA MIMIC just discussed, a version that adds some residual covariances among the indicators, and MIMIC with a bifactor measurement model where the general factor is also regressed on the covariates.

ESEM MIMIC Examples

- User's Guide ex 5.24: ESEM
  - f1-f2 BY y1-y8 (*1); f1-f2 ON x1-x2;
- Asparouhov-Muthén (2026): PSEM SEFA for HS19 with second-order factor regressed on grade and gender
- Muthén (2025): ESEM and PSEM with WLSMV for binary factor indicators of antisocial behavior
  - ESEM MIMIC analysis: EFA measurement model with factors regressed on covariates and covariates influencing indicators directly

Slide 70 lists some applications of ESEM MIMIC.

Apart from the User's Guide example 5.24, our 2026 paper illustrates the use of a SEFA measurement model in a PSEM analysis where the second-order factor is regressed on covariates.

The Muthén 2025 handbook chapter shows ESEM together with PSEM with WLSMV estimation for binary indicators. Both the factors and the indicators are regressed on covariates. The indicator regressions capture measurement non-invariance. This example will be shown next.

Input for ESEM MIMIC with 17 Items, 3 Factors, and 11 Covariates: ASB Example (Muthén, 2025)

```
VARIABLE:
              CATEGORICAL = property-gambling;

ANALYSIS:     ESTIMATOR = WLSMV;

MODEL:        ! 3-factor EFA measurement model:
              f1-f3 BY property-gambling (*1);

              ! EFA factors regressed on covariates:
              f1-f3 ON sex black hisp single divorce dropout
              college onset age94 dep abuse;

OUTPUT:       TECH1 TECH4 TECH10
              STANDARDIZED RESIDUAL;

PLOT:         TYPE = PLOT3;
```


Slide 71 shows the input used in the 2025 paper. The example refers to antisocial behavior where 17 binary variables are used as indicators of 3 factors regressed on 11 covariates.

The VARIABLE command specifies the indicators as categorical and the ANALYSIS command requests weighted least squares estimation. The EFA specification uses the ESEM approach, followed by the regression of the EFA factors on the covariates and using the 2 rotation steps outlined earlier on slide 68.

ESEM-PSEM of ASB

- ESEM-PSEM analysis: Direct effects added from covariates to factor indicators to capture measurement non-invariance
  - Using PSEM ALF priors makes it possible to identify all direct effects
  - The ALF prior variance is chosen so that the loglikelihood is the same as the "null" model which in this case is ESEM with regressions of all indicators on all covariates but no regressions of the factors on the covariates
- 17 factor indicators and 11 covariates results in 187 possible direct effects, of which PSEM finds that 40 are significant
  - Freeing these 40 direct effects changes the relations between the factors and the covariates

Slide 72 discusses the addition of direct effects from the covariates to the indicators. The presence of such effects reflect measurement non invariance.

The direct effects are not identified when added all at the same time, but PSEM using ALF makes it possible to identify them.

Here, it is important to choose the prior variance of ALF properly. It should be chosen such that the loglikelihood is the same as the so called null model which in this case is the model with all direct effects but no effects on the factors.

There is a total of 187 possible direct effects. The PSEM ALF approach avoids a long stepwise search for significant effects and includes all of them in one single run. In this case, 40 effects were significant. These effects can then be freed in a new run to check to which extent the relations have changed between the factors and the covariates. The paper, which is on the Mplus website, gives further details and discusses the interpretation of the findings.

The input is shown on the next slide.

Input for PSEM Direct Effects for ASB

```
                    USEVARIABLES = property-gambling sex black hisp single
                    divorce dropout college onset age94 dep abuse;
                    CATEGORICAL = property-gambling;

ANALYSIS:           ESTIMATOR = WLSMV;

MODEL:              f1-f3 BY property-gambling (*1);
                    f1-f3 ON sex black hisp
                    single divorce dropout
                    college onset age94 dep abuse;
                    property-gambling ON sex-abuse (p1-p187);

MODEL PRIORS:       p1-p187 ALF(0,1);

OUTPUT:             TECH1 TECH4 TECH10
                    STANDARDIZED RESIDUAL;

PLOT:               TYPE = PLOT3;
```


Slide 73 gives the input for the analysis just discussed. The bolded lines show the changes made to the regular ESEM MIMIC. The direct effects are labeled and then given ALF priors with mean zero and variance 1.

3.2 EFA/CFA on EFA/CFA

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - **EFA/CFA on EFA/CFA**
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- EFA theory

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Slide 74 returns to the outline of the presentation. While MIMIC is a structural equation model, we are now turning to more general SEM settings with various combinations of EFA and CFA measurement models.

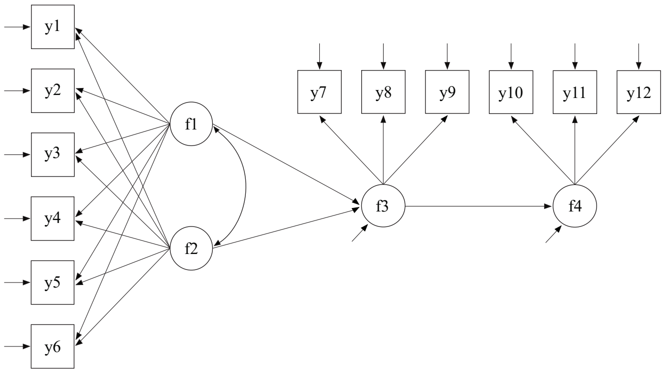
EFA/CFA On EFA/CFA

- CFA on EFA
- CFA on EFA and X and Y
- EFA on CFA
- EFA on CFA and X and Y
- EFA on EFA

- Factors of an EFA must be in a block by themselves, but you can have several EFA blocks

Slide 75 lists some variations of combinations of EFA and CFA measurement models in structural equation models that can be handled by Mplus. For example, there can be a CFA model for the dependent variable side of an SEM with EFA factors on the independent variable side. The reverse is also possible. Other independent or dependent variables can be added - here marked as X and Y. We will also see examples of EFA factors regressed on EFA factors, either in two different EFA blocks or within the same block. Some examples will clarify this.

CFA Regressed On EFA: UG Ex5.25




```
TITLE:     this is an example of SEM with EFA and
           CFA factors with continuous factor indicators
DATA:      FILE IS ex5.25.dat;
VARIABLE:  NAMES = y1-y12;
MODEL:     f1-f2 BY y1-y6 (*1); ! EFA factors
           f3 BY y7-y9; ! CFA factor
           f4 BY y10-y12; ! CFA factor
           f3 ON f1-f2;
           f4 ON f3;
```

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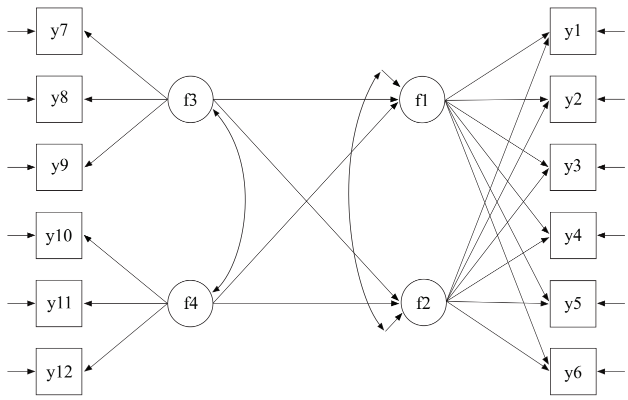
Slide 76 shows an example of CFA factors regressed on EFA factors. This is the Mplus User's Guide example 5.25. The MODEL command shows that the f1, f2 factors are part of an EFA block, while f3 and f4 are CFA factors.

EFA Regressed On CFA: UG Ex5.25 Reversed

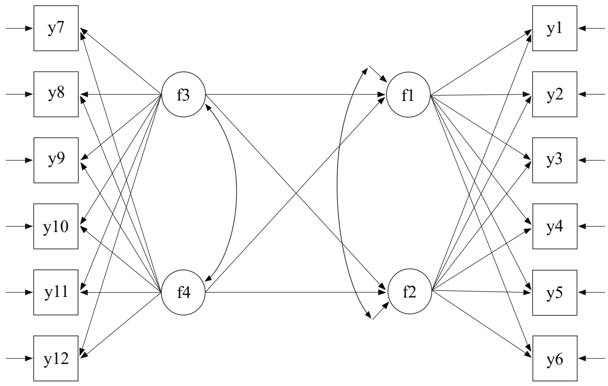


MODEL:

```
f1-f2 BY y1-y6 (*1); ! EFA factors
f3 BY y7-y9; ! CFA factor
f4 BY y10-y12; !CFA factor
f1-f2 ON f3 f4;
```


Slide 77 shows the reverse situation with an example of EFA factors regressed on CFA factors. The EFA factors f1 and f2 are now on the dependent variable side of the model.

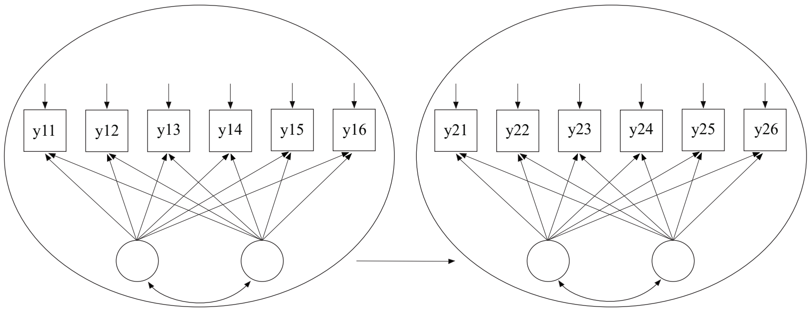
EFA Regressed On EFA: UG Ex5.25 Modified: Two EFA Blocks



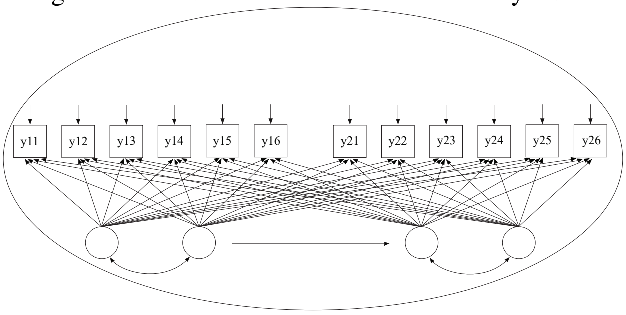
MODEL:

```
f1-f2 BY y1-y6 (*1);
f3-f4 BY y7-y12 (*2);
f1-f2 ON f3 f4;
```


Slide 78 shows an example of EFA factors regressed on EFA factors. This is an example of using two EFA blocks. Here, f1 and f2 are in EFA block 1, designated by (*1), while f3 and f4 are in EFA block 2 designated by (*2).



Regression between 2 blocks: Can be done by ESEM



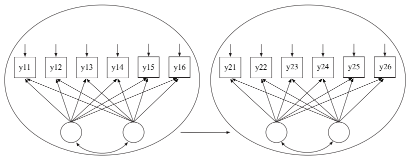
Regression within 1 block: Can only be done by PSEM

Slide 79 shows the difference between using ESEM and using PSEM. The top part of the slide shows regression between two different EFA blocks just as in the example of the previous slide. This can be done by ESEM.

The bottom part of the slide shows that all four factors influence all 12 indicators. This represents a single EFA block with a regression between the EFA factors. Such a within-block regression can only be carried out with the help of PSEM.

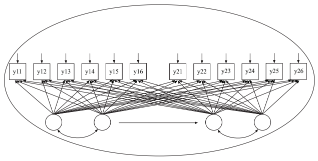
The corresponding two inputs are shown on the next slide.

ESEM vs PSEM Input



Regression between 2 blocks: Can be done by ESEM

```
f1-f2 BY y11-y16 (*1);
f3-f4 BY y21-y26 (*2);
f3-f4 ON f1-f2;
```



Regression within 1 block: Can only be done by PSEM

```
            f1-f4 BY y11-y26* (p1-p48); f1-f4@1;
            f3-f4 ON f1-f2;
MODEL PRIORS:
            p1-p48~GEOMIN(4,1,0.0001);
```

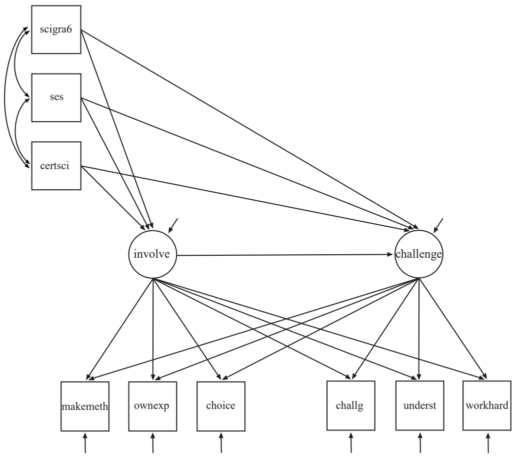

Slide 80 shows the input for the two situations just discussed.

For the top case, the input shows two EFA blocks expressed in ESEM terms.

For the bottom case, a single EFA block is specified using the PSEM GEOMIN approach to EFA - which was one of the 3 EFA approaches we listed on slide 8 and used on slide 14.

The MODEL PRIORS command specifies GEOMIN rotation for the set of 4 factors. The prior variance is set at 1 and a small epsilon value is chosen. The GEOMIN prior settings are shown in the EFA Theory section of the presentation.

EFA in SEM Using PSEM: Science Example




```
MODEL:
        involv BY makemeth*1 ownexp*1 choice*1 challg*0 underst*0 workhard*0 (p1-p6);
        chall BY makemeth*0 ownexp*0 choice*0 challg*1 underst*1 workhard*1 (p7-p12);
        involv@1 chall@1; ! 1 block: the 2 factors measured by makemeth-workhard
        chall ON involv; involv ON scigra6-certsci; chall ON scigra6-certsci*0;
ANALYSIS:
        ! ITERATIONS = 10000; STARTS = 50; CONVERGENCE = 0.000001;
MODEL PRIORS:
        p1-p12~GEOMIN(2,1,0.0001);
```

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Slide 81 shows an example of regression within an EFA block. The example is based on the Science data used in Muthén & Asparouhov (2012). A subset of this was used as a PSEM illustration in the Asparouhov (2023) M3 workshop which you find on the next slide.

The two EFA factors challenge and involve are measured by six indicators. The challenge factor is regressed on the involve factor, that is, the model has a within-block regression like at the bottom of slide 80. Both factors are regressed on three covariates.

The input gives labels to the 12 loadings which are given GEOMIN priors. The number 2 refers to the number of factors, 1 is the prior variance, and epsilon is set to a small number which is required in this situation.

PSEM Presentations and Web Site

- Asparouhov & Muthén (2024). Penalized structural equation models. Structural Equation Modeling. 31, 429-454.
- Asparouhov & Muthén (2025). Methodological advances with penalized structural equation models. Structural Equation Modeling. 32, 688-716.
- Asparouhov (2023) M3 conference workshop: `https://www.statmodel.com/download/M3TeachingSlides.pdf`
- Asparouhov (2023) M3 conference talk: `https://www.statmodel.com/download/M3Talk.pdf`
- All of the above found at the web site Special Mplus Topics, PSEM: `https://www.statmodel.com/psem.shtml`

Slide 82 gives a list of references for further PSEM readings. As you see at the bottom, there is a special web site for PSEM.

4 EFA in a multiple-group setting

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- **EFA in a multiple-group setting**
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- EFA theory

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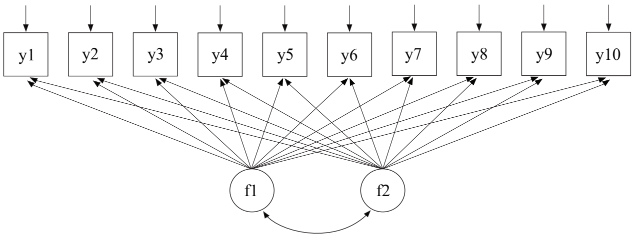
Slide 83 returns to the Outline of the presentation. After having covered EFA in an SEM setting, we are now turning to EFA in the common case of a multiple-group setting. This will also lead to discussing the useful technique of Alignment.

ESEM for Multiple-Group EFA: UG Ex 5.27 Part 3



TITLE:	this is an example of multiple-group EFA with continuous factor indicators with scalar measurement invariance
DATA:	FILE IS ex5.27.dat;
VARIABLE:	NAMES = y1-y10 group; GROUPING = group (1 = g1 2 = g2);
ANALYSIS:	MODEL = CONFIGURAL METRIC SCALAR;
MODEL:	f1-f2 BY y1-y10 (*1); ! scalar invariance

- Configural and metric invariance specifications shown in UG ex 5.27 part 1 and part 2

Slide 84 shows an Mplus User's Guide example where there are 2 EFA factors in 2 groups. The MODEL command specifies scalar invariance, that is, both loadings and indicator intercepts are held equal across the 2 groups. In the ANALYSIS command, however, we request testing of all 3 relevant measurement models by the specification model = configural metric scalar. How to specify configural and metric measurement models in the MODEL command is shown in parts 1 and 2 of the User's Guide example.

MODEL FIT INFORMATION

Invariance Testing

Model	Number of Parameters	Chi-Square	Degrees of Freedom	P-Value
Configural	78	63.815	52	0.1261
Metric	62	77.687	68	0.1975
Scalar	54	82.326	76	0.2900

Models Compared	Chi-Square	Degrees of Freedom	P-Value
Metric against Configural	13.872	16	0.6083
Scalar against Configural	18.511	24	0.7775
Scalar against Metric	4.640	8	0.7953

Slide 85 shows the output for the invariance testing obtained by model = configural metric scalar. This is simulated data and all 3 models fit well. With real data, the comparison of models at the bottom part of the output is helpful for checking at which point misfit shows up. Metric invariance, that is, invariant loadings, may fit sufficiently well, but often scalar invariance does not fit well with its added requirement of invariant intercepts for the factor indicators.

4.1 Alignment

Alignment

- Metric and scalar measurement invariance often rejected
- The alignment model has the same fit as the configural model
- Alignment minimizes the amount of measurement noninvariance in intercepts and loadings by estimating group-varying factor means and variances
- The group-varying factor means and variances are not identified in the configural model - alignment avoids this problem by adding the necessary extra information via optimization of a simplicity criterion similar to EFA rotation criteria avoiding indeterminacies

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Slide 86 turns to the topic of Alignment which is a very useful technique in multiple-group studies with both CFA and EFA measurement models.

The premise for Alignment is that metric and scalar measurement invariance is often rejected.

A solution to this is the fact that the alignment model only requires that a configural model fits well, that is, the pattern of zero loadings is invariant across groups.

Simply put, alignment minimizes the amount of measurement noninvariance in intercepts and loadings by estimating group-varying factor means and variances.

The interesting thing is that group-varying factor means and variances are not identified in the configural model. Alignment avoids this problem by adding the necessary extra information via optimization of a simplicity criterion similar to how EFA rotation criteria avoid the indeterminacies.

Alignment Papers

- Multiple-group alignment:
  - Asparouhov & Muthén (2014). Multiple-group factor analysis alignment. Structural Equation Modeling: A Multidisciplinary Journal, 21:4, 495-508.
  - Muthén & Asparouhov (2014). IRT studies of many groups: The alignment method. Frontiers in Psychology
  - Muthén & Asparouhov (2018). Recent methods for the study of measurement invariance with many groups: Alignment and random effects. Sociological Methods & Research, 47:4 637-664.
- Generalized multiple-group alignment - ASEM, AESEM (allowing cross-loadings, ESEM, factors regressed on factors, covariates):
  - Asparouhov & Muthén (2023). Multiple group alignment for exploratory and structural equation models. Structural Equation Modeling: A Multidisciplinary Journal, 30(2), 169-191.
  - Asparouhov & Muthén (2025). Alignment for multiple group second-order factor analysis. Mplus Web Notes: No. 27. Version 1. December 26, 2025.
- `https://www.statmodel.com/MeasurementInvariance.shtml`

Slide 87 gives a list of Alignment papers to study. They can all be found on the Special Mplus Topics page Measurement Invariance, which has many application papers.

Alignment Theory Briefly Stated: 2 Steps

- 1. Estimate the configural model:
  - Loadings ($\lambda_{configural}$) and intercepts ($\nu_{configural}$) free across groups, factor means fixed at zero in all groups, factor variances fixed at 1 in all groups
- 2. Do the alignment optimization:
  - Free the factor means and variances and choose their values to minimize the amount of noinvariance using a simplicity function
- In step 2, the factor means α_j and variances ψ_j are free parameters to be estimated, maintaining the configural model fit while obtaining the aligned λ_j and ν_j for group j:

$$\lambda_j = \lambda_{j,configural} / \sqrt{\psi_j} \tag{1}$$
$$\nu_j = \nu_{j,configural} - \alpha_j \, \lambda_{j,configural} / \sqrt{\psi_j} \tag{2}$$

It is not our intention to fully discuss Alignment in this presentation, but slide 88 gives the basic idea of the steps taken in this technique.

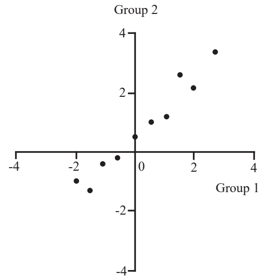
In Step 1, the configural model is estimated. Here, loadings and intercepts are free across groups, factor means are fixed at zero in all groups, and factor variances are fixed at 1 in all groups. This is the same specification as estimating each group separately.

Step 2 does the alignment optimization. Here, alignment frees the factor means and variances and choose their values to minimize the amount of noninvariance using a simplicity function.

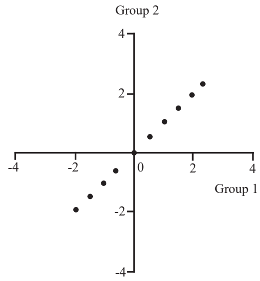
Equations 1 and 2 show the aligned loadings (λ) and indicator intercepts (ν) as functions of the configural counterparts from step 1 and the free factor variances and factor means from step 2.

Why "Alignment"? Intercept Invariance but Factor Diffs

- Intercepts of 10 indicators (dots)
- 1 factor
- 2 groups (axes)
- Factor means = 0, -1
- Factor variances = 1, 2



Unaligned: Configural model (factor mean=0, factor variance=1 in both groups)



Aligned: Taking into account the group differences in factor means and variances

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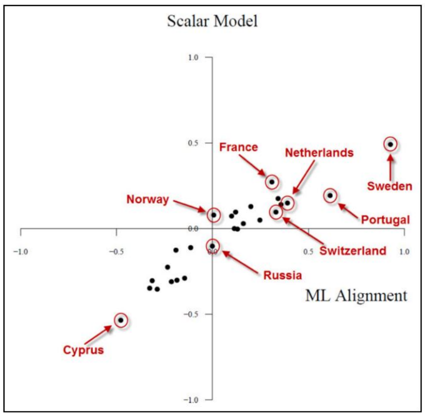
Slide 89 explains why the term alignment is used. The figures show 10 factor indicators measuring 1 factor in 2 groups. The intercepts of the indicators are the same in the 2 groups but the groups have different factor means and variances.

The top figure shows the first step of alignment where the factor means are set to zero in both groups and the factor variances are set to 1 in both groups.

The bottom figure shows the intercepts after alignment. Group differences in factor means and variances are allowed which makes the intercepts be perfectly aligned along the 45-degree line showing that they are the same in the 2 groups.



Factor means for 26 countries: Scalar versus Alignment

- Alignment agrees with Scalar that Sweden/Cyprus have the highest/lowest level of tradition nonconformity
- Alignment disagrees with Scalar regarding the difference between Portugal and Netherlands, between France and Switzerland, as well as between Norway and Russia

Slide 90 shows an application to a 26-country analysis where tradition nonconformity is measured. The figure shows the resulting factor means as obtained by alignment on the x-axis and by using a scalar model on the y-axis. As is often found, the two approaches do not give widely different results - the factor means largely fall on the 45-degree line. But, the two approaches do show some differences in the comparison of countries. For example, Alignment agrees with Scalar that Sweden/Cyprus have the highest/lowest level of tradition nonconformity. But Alignment disagrees with Scalar regarding the difference between Portugal and Netherlands in that these countries show a bigger difference as judged by the alignment axis than as judged by the scalar axis. Likewise, there is disagreement in the comparison between France and Switzerland, as well as between Norway and Russia.

Input for Alignment in the Single-Factor Case

```
DATA:        FILE IS ess05Traco.dat;
VARIABLE:    NAMES = country essround ipfrule ipmodst ipbhprp imptrad;
             USEVARIABLES = ipmodst imptrad ipfrule ipbhprp;
             MISSING = ipfrule-imptrad (7-9);
             GROUPING = country(2-18 21-28 30);

ANALYSIS:    ESTIMATOR = ML;
             ALIGNMENT = FIXED(22);

MODEL:       traco BY ipmodst-ipbhprp;

OUTPUT:      ALIGN; ! providing extra alignment output

PLOT:        TYPE = PLOT2;
```

- STANDARD ERROR COMPARISON INDICATES THAT THE FREE ALIGNMENT MODEL MAY BE POORLY IDENTIFIED. USING THE FIXED ALIGNMENT OPTION MAY RESOLVE THIS PROBLEM.
- Muthén & Asparouhov (2018): Tradition-conformative measures in 26 European countries

Slide 91 shows the input for the 26-country example discussed earlier. In the VARIABLE command, the GROUPING option specifies the countries involved.

ALIGNMENT is specified in the ANALYSIS command. Typically, the alignment uses the FREE option where no specific group is used as basis for the comparison. In this case, however, this gives the error message shown at the bottom:

STANDARD ERROR COMPARISON INDICATES THAT THE FREE ALIGNMENT MODEL MAY BE POORLY IDENTIFIED. USING THE FIXED ALIGNMENT OPTION MAY RESOLVE THIS PROBLEM.

With the FIXED option, a group with the smallest factor mean may be chosen as the basis for comparison. In this case, country 22 was chosen. Using FIXED without a qualifier uses the first group for comparison.

This example has only 1 factor which means that EFA is not relevant. The next example shows alignment with a 4-factor EFA.

AESEM: Alignment with ESEM

- Asparouhov & Muthén (2023): PISA example
  - 15-year old students, 30 OECD countries/groups, N = 249,840
  - Sampling weights, PSUs: Complex survey data (Asparohov, 2005)
  - 22 factor indicators, 4 factors, 3 covariates
- 4 Models:
  - No alignment, scalar invariance:
    - (1) SEM: CFA
    - (2) ESEM: EFA
      - More parameters than (1) due to cross-loadings
  - Alignment, configural invariance:
    - (3) ASEM: CFA
      - More parameters than (1) due to configural
    - (4) **AESEM**: EFA
      - More parameters than (1) due to cross-loadings and configural

Slide 92 shows a large example from the PISA study (Programme for International Student Assessment) which is an international study that evaluates the knowledge and skills of 15-year-old students in reading, mathematics, and science. The sample consists of about 250 thousand students from 30 countries. The data features sampling weights, PSUs (primary sampling units) and school clusters, that is, complex survey data features. The implementation of such features is discussed in Asparouhov (2005).

The dataset considered here has 22 indicators of 4 factors. In addition, 2 covariates are added to the alignment analysis. Four models are compared in the Asparouhov-Muthén (2023) paper:

No alignment, scalar invariance:
(1) SEM: CFA
(2) ESEM: EFA
- We note that (2) has more parameters than (1) due to cross-loadings

Alignment, configural invariance:
(3) ASEM: CFA
- We note that (3) has more parameters than (1) due to configural
(4) AESEM: EFA
- We note that (4) has more parameters than (1) due to cross-loadings and configural

The question is which of the 4 models is best for the PISA data.

Comparison of Multiple-Group Analyses for PISA Example

- Table 9 in Asparouhov & Muthén (2023): AESEM fits best

Model	SEM	ESEM	ASEM	**AESEM**
Number of parameters	1,476	1,530	2,520	4,140
Chi-square	162,449	145,919	85,308	62,943
Degrees of freedom	8,754	8,700	7,710	6,090
BIC	10,404,759	10,376,631	10,286,767	**10,268,659**
CFI	0.92	0.93	0.96	0.97
TLI	0.92	0.93	0.95	0.96
SRMR	0.049	0.045	0.029	0.019
RMSEA	0.046	0.044	0.035	0.034

- SEM: scalar invariance, CFA
- ESEM: scalar invariance, EFA.
- ASEM: alignment (configural), CFA.
- AESEM: alignment (configural), EFA

Slide 93 compares the 4 approaches shown as different columns in the table. You see the increase in parameters going from left to right with AESEM having over 4 thousand parameters. Chi-square is difficult to work with here given the large sample size of 250 thousand students. CFI improves going from left to right as expected with increasing number of parameters. BIC is useful here because it has a penalty for the number parameters. BIC points to AESEM as the best model with the lowest BIC. AESEM uses configural invariance for an EFA measurement model.

The AESEM input is shown on the next slide.

Input for AESEM Analysis of the PISA Example Asparouhov & Muthén (2023)

```
DATA:        FILE = pisa06_alignment_final_data_r.dat;

VARIABLE:    NAMES = schoolid stidstd country oecd w_fstuwt st16q01-st16q05
             st17q01-st17q08 st18q01-st18q10 st19q01-st19q06 st21q01-st21q08
             st29q01-st29q04 st35q01-st35q05 st37q01-st37q06 pv1scie gender ses
             cntgen zpvliscie zgenders zses;
             USEVARIABLES = st16q01-st16q05 st35q01-st35q05 st29q01-st29q04
             st17q01-st17q08 gender zses zpvliscie;
             WEIGHT = w_fstuwt;
             CLUSTER = schoolid;
             MISSING = .;
             GROUPING = country(30);

DEFINE:      schoolid=(country*10000)+schoolid; ! creates unique schoolid

ANALYSIS:    TYPE = COMPLEX;
             ALIGNMENT = FIXED; ! group 1 chosen for comparison
             STARTS = 30; ASTARTS = 100; ! starts for step 1 and step 2 (alignment)

MODEL:       f1-f4 BY st16q01-st17q08 (*1);
             f1-f4 ON gender zses zpvliscie;
```


Slide 94 shows the input for the AESEM analysis just discussed. It specifies that there are 30 countries. TYPE = COMPLEX handles the sampling weights and the clusters. The fixed alignment option is chosen here which implies that the factor means/intercepts are fixed at zero in the first group. Given the complexity of the analysis, 30 sets of random starting values are used together with 100 alignment starts.

The modeling uses an ESEM setup for the 4 factors behind the 22 indicators. The 4 factors are regressed on 3 covariates.

This is a time-consuming analysis given the large number of parameters.

The results are discussed in Section 4 of Asparouhov & Muthén (2023).

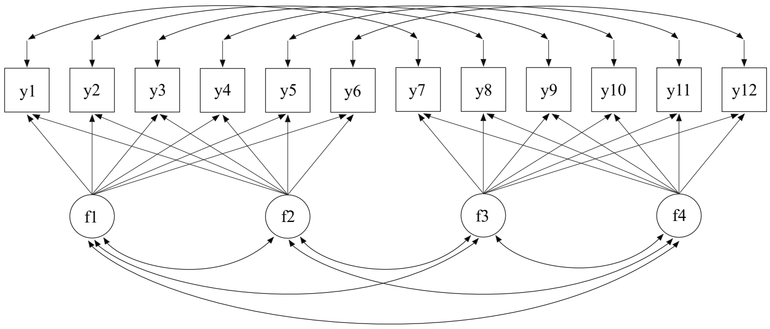
5 EFA in a longitudinal setting

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- **EFA in a longitudinal setting**
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- EFA theory

Slide 95 returns to the Outline for the presentation. We have now come to the final setting for EFA, namely longitudinal analysis.

ESEM for Longitudinal EFA: UG Ex 5.26




```
TITLE:      this is an example of an EFA at two time
            points with factor loading invariance and
            correlated residuals across time

DATA:       FILE = ex5.26.dat;

VARIABLE:   NAMES = y1-y12;

MODEL:      ! two EFA blocks t1 and t2 with
            ! metric invariance specified by using 1:
            f1-f2 BY y1-y6 (*t1 1);
            f3-f4 BY y7-y12 (*t2 1);
            y1-y6 PWITH y7-y12;
```

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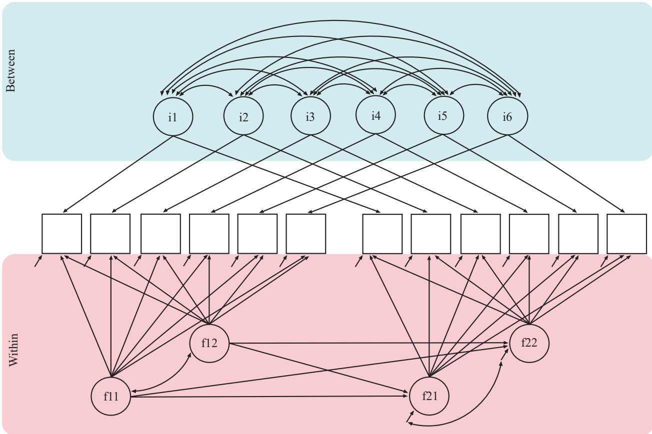
Slide 96 shows the User's Guide example 5.26 where you have two timepoints. The same set of 6 indicators are measured at the two timepoints. It is assumed that we have metric invariance, that is, factor loading invariance across time.

As the MODEL statement shows, this example uses two EFA blocks labeled t1 and t2, each block having an EFA with 2 factors. The label "1" indicates metric invariance. The indicator residuals are pairwise correlated using PWITH: y1 with y7, y2 with y8, etc. This may be necessary given that they represent the same indicator at the two timepoints.

An Alternative to UG Ex 5.26 Longitudinal EFA



- Random intercepts for each indicator (between person variables). The random intercepts are correlated and can be given a factor structure that is different from the within structure
- Within factors (within person variables) with auto regressions
- Auto regressions among indicator-specific residuals (not shown)

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Slide 97 shows an alternative specification for longitudinal EFA. It emphasizes the distinction between within-person and between-person factors and is in line with models such as random-intercept cross-lagged panel data (RI-CLPM). This alternative is suitable when there are more than 2 timepoints - only 2 are shown here.

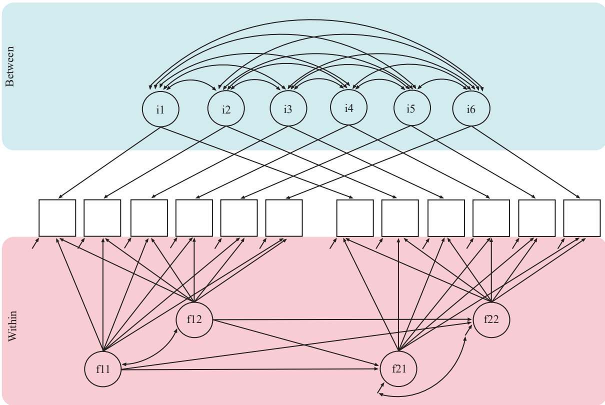
As seen in the figure, each indicator has a random intercept factor - its loadings are fixed at 1 - that influences the indicator at all timepoints, thereby creating a correlation that is constant across time for the indicator. This intercept varies across individuals and can be seen as a stable trait.

The indicators also correlate across time via the factors which vary across time. These factors are regressed on the factors at the previous timepoint.

The random intercepts can either be freely correlated as shown in the figure or given a factor specification. This then leads to the distinction of between-person and within-person factors. For example, there may be one factor on between and 2 on within.

In addition, the residuals of the indicators can have auto-regressions across time. This is left out of the figure to not clutter it.

Relation of Longitudinal EFA to RI-CLPM with Multiple Indicators



- The figure corresponds to one multidimensional process in an RI-CLPM
- A second process can be added so that cross-lagged relations between the factors can be studied
- Unlike regular RI-CLPM, EFA factors are allowed (one process can be EFA and the other CFA)

Slide 98 points out the relation to RI-CLPM when RI-CLPM has multiple indicators. The figure is the same as on the previous slide and can now be seen as one half of a two-process RI-CLPM.

In an RI-CLPM of this kind there are two processes that each has 2 factors per timepoint and each has an EFA measurement model. It is also possible that one of the two processes has a CFA measurement model.

Example: Longitudinal Exploratory Factor Analysis of Positive Affect Across 7 Week Days

- PA average per day for 7 days (Tue - Mon): N = 244, T = 7
  - Dietvost et al. (2021). Grumpy of depressed? Disentangling typically developing adolescent mood from prodromal depression using experience sampling methods. Journal of Adolescence.
- PA is the average of 6 items:
  - Low arousal: Relaxed, satisfied, confident
  - High arousal: Happy, energetic, excited
- What's the factor structure for the 6 items?
  - EFA is needed
- Separate analyses of each day has disadvantages:
  - Using less information
  - Confounding trait with state variation
  - Cattell-Molenaar-Hamaker-Steyer-Eid-Geiser (LST theory)

Slide 99 shows an example of a longitudinal study of positive affect (PA) across 7 week days. PA is measured by 6 items. While the average of the items is typically used, we will instead analyze them on the item level. There are 3 low arousal items and 3 high arousal items. To not presume that the 6 items measure a single factor, we will apply an EFA with 1-3 factors.

The longitudinal EFA has several advantages over separate factor analyses of each of the 7 days in that it uses more information and does not confound trait with state variation in line with the literature of Cattell-Molenaar-Hamaker-Steyer-Eid-Geiser (LST theory).

The input for this example is shown on the next slide.

Input for Longitudinal ESEM Factor Analysis of PA Items

```
              USEVARIABLES = relax1-excit7;
              ! 6 PA items, 3 low arousal 3 high arousal:
              ! relaxed (pala1) satisfied (pala2) confident (pala3)
              ! happy (paha1) energetic (paha2) excited (paha3)
              ! 7 time points

ANALYSIS:     ESTIMATOR = MLR;

MODEL:        ! random intercepts for the 6 items
              i1 BY relax1-relax7@1;
              i2 BY satis1-satis7@1;
              i3 BY conf1-conf7@1;
              i4 BY happy1-happy7@1;
              i5 BY energ1-energ7@1;
              i6 BY excit1-excit7@1;

              ! auto-regressions among factor indicators residuals:
              relax2^-relax7^ PON relax1^-relax6^ (ar1);
              satis2^-satis7^ PON satis1^-satis6^ (ar2);
              conf2^conf7^ PON conf1^-conf6^ (ar3);
              happy2^-happy7^ PON happy1^-happy6^ (ar4);
              energ2^-energ7^ PON energ1^-energ6^ (ar5);
              excit2^-excit7^ PON excit1^-excit6^ (ar6);
```


Slide 100 shows the input for the longitudinal EFA using ESEM.

The MODEL command first specifies 6 random intercepts with loadings fixed at 1.

Next, first-order auto-regressions among the residuals of the factor indicators are specified.

The input continues on the next slide.

Longitudinal ESEM Factor Analysis Input Cont'd

```
! 2-factor ESEM with metric (loading) invariance as in UG ex 5.26
! (factor variances at first time point are automatically fixed at 1):
f11-f12 BY relax1 satis1 conf1 happy1 energ1 excit1(*1 1);
f21-f22 BY relax2 satis2 conf2 happy2 energ2 excit2(*2 1);
f31-f32 BY relax3 satis3 conf3 happy3 energ3 excit3(*3 1);
f41-f42 BY relax4 satis4 conf4 happy4 energ4 excit4(*4 1);
f51-f52 BY relax5 satis5 conf5 happy5 energ5 excit5(*5 1);
f61-f62 BY relax6 satis6 conf6 happy6 energ6 excit6(*6 1);
f71-f72 BY relax7 satis7 conf7 happy7 energ7 excit7(*7 1);

! auto-regressions among factors to reduce the number of parameters
f21-f22 ON f11-f12; f31-f32 ON f21-f22; f41-f42 ON f31-f32;
f51-f52 ON f41-f42; f61-f62 ON f51-f52; f71-f72 ON f61-f62;
i1-i6 WITH f11-f72@0;

! scalar invariance for intercepts:
[relax1-relax7] (int1);
[satis1-satis7] (int2);
[conf1-conf7] (int3);
[happy1-happy7] (int4);
[energ1-energ7] (int5);
[excit1-excit7] (int6);
[f11-f12@0 f21-f72*];
```


Slide 101 continues the input from the previous slide.

First, a 2-factor ESEM is specified for each of the 7 timepoints. This uses 7 different EFA blocks. The label "1" indicates metric invariance across the timepoints (blocks) in the same way as for multiple-group ESEM discussed earlier.

Second, the factors are regressed on themselves at the previous timepoint. The alternative of freely correlating all factors leads to many more parameters.

Third, scalar invariance is added to the metric specification by applying equality constraints for the intercepts of the 6 indicators. This addition is not necessary for the purposes of this analysis.

Standardized Factor Loading Estimates for Metric Model

	Estimate	S.E.	Est./S.E.	Two-Tailed P-Value
F11 BY				
RELAX1	**0.471**	0.052	9.072	0.000
SATIS1	**0.550**	0.061	8.999	0.000
CONF1	**0.352**	0.055	6.379	0.000
HAPPY1	**0.274**	0.046	5.984	0.000
ENERG1	0.016	0.030	0.540	0.589
EXCIT1	-0.007	0.002	-3.331	0.001
F12 BY				
RELAX1	0.001	0.017	0.078	0.938
SATIS1	-0.007	0.023	-0.312	0.755
CONF1	0.023	0.037	0.623	0.533
HAPPY1	**0.308**	0.051	6.089	0.000
ENERG1	**0.475**	0.053	8.901	0.000
EXCIT1	**0.613**	0.055	11.107	0.000
F11 WITH F12				
	0.616	0.094	6.573	0.000

- Unlike EFA, ESEM analyzes a sample covariance matrix - although the factors at t=1 have variance 1, the outcomes don't: Stand'd values correspond to EFA

Slide 102 shows the two-factor EFA loadings for the 6 PA items. The two factors correspond largely to low versus high arousal items. The Happy item deviates from this in that it loads equally on both factors. The factors correlate 0.6. The correlation is significantly different from 1, rejecting the single-factor model and therefore calls into question the use of a single average score for PA.

Further Results

- Factor indicator-specific residual AR1:
  - Significant for 3 out of the 6 items: Confident, happy, energetic
- Lag1 regressions among the two within factors:
  - AR1 significant for only high-arousal PA factor
  - Insignificant cross-lagged effects between the two factors
- Percentage variance explained by the random intercepts
- Variance decompositions, reliability:
  - Eid et al. (2017). On the definition of latent-state-trait models with autoregressive effects. European Journal of Psychological Assessment, 33, 285-295

Slide 103 shows further results.

The factor indicator-specific residual AR1s are significant for 3 out of the 6 items: Confident, happy, energetic.

Regarding lag1 regressions among the two within factors, AR1 is significant for only the high-arousal PA factor, while there are insignificant cross-lagged effects between the two factors.

Other things to look for are percentage variance explained by the random intercepts and the variance decompositions, including reliability.

A relevant reference here is Eid et al. (2017). On the definition of latent-state-trait models with autoregressive effects. European Journal of Psychological Assessment, 33, 285-295.

Adding Other Variables

- Validating the need for two factors: Is there an important difference between how the two factors relate to other variables?
  - Do they have different relations to covariates or distal outcomes?
  - Such other variables can be included in longitudinal ESEM
- The time-varying covariate tired is included in the current example
- Further results are discussed in the 2023 M3 workshop slides `https://www.statmodel.com/download/Muthen2023M3Workshop.pdf`

Slide 104 brings up validation of the two EFA factors. - Is there an important difference between how the two factors relate to other variables? - Do they have different relations to covariates or distal outcomes?

Such other variables can be included in longitudinal ESEM. The time-varying covariate tired is included in the current example. Further results are discussed in the Muthén 2023 M3 workshop slides

https://www.statmodel.com/download/Muthen2023M3Workshop.pdf

The next 2 slides discuss some more technical aspects of the longitudinal EFA.

ESEM with Metric/Scalar Invariance as Part of a Bigger Model: How Does it Work?

- The unrotated model is estimated holding the unrotated loadings equal across time, fixes factor variances at 1 for the first time point, and uses uncorrelated factors. After rotation, this identifies factor covariance at the first time point, and unrestricted factor covariance matrices at all other time points (2 factors involve 2 par's decided by rotation: TECH1 shows 2 more par's than in the results). Asparouhov & Muthén (2009)
- The unrotated factors can have an AR model
- The rotation transformation is then applied to the entire model
- The necessary restriction is that if a variable is regressed on a factor in an EFA block it has to be regressed on all factors in the EFA block - the same applies for the opposite regression
- CFA factors (like random intercepts) can be combined with EFA factors
  - If one EFA factor is correlated with the CFA factor all other EFA factors in the same block must also be correlated (random intercepts are uncorrelated with the ESEM factors)

Slide 105 shows how longitudinal EFA works using the ESEM framework of Asparouhov & Muthén (2009).

As usual, ESEM starts with an unrotated model. The unrotated model is estimated holding the unrotated loadings equal across time, fixes factor variances at 1 for the first time point, and uses uncorrelated factors. After rotation, this identifies factor covariance at the first time point, and unrestricted factor covariance matrices at all other time points (2 factors involve 2 par's decided by rotation: TECH1 shows 2 more par's than in the results).

The unrotated factors can have an AR model. The rotation transformation is then applied to the entire model.

The necessary restriction is that if a variable is regressed on a factor in an EFA block it has to be regressed on all factors in the EFA block - the same applies for the opposite regression.

CFA factors (like random intercepts) can be combined with EFA factors.

If one EFA factor is correlated with the CFA factor all other EFA factors in the same block must also be correlated (random intercepts are uncorrelated with the ESEM factors).

Longitudinal Factor Analysis Chi-Square Testing: Sample Size, Number of Variables, Number of Time Points

- P factor indicators per time point and T time points result in:
  - P*T variables = 6*7 = 42 in our example
  - The H0 model may have more parameters than the sample size
  - The H1 model has P*T*(P*T+1)/2 = 903 parameters in our example (N = 244)
- What is the quality of the regular chi-square testing of the H0 model against the unrestricted H1 model?
- Simulations based on the estimated model suggest inflated chi-square 5% reject proportions:
  - N = 250: 0.41
  - N = 500: 0.14
  - N = 1000: 0.11
  - Parameter estimates, SEs, and coverage good even at smaller N
- Small sample sizes may not be able to handle large P*T: N > P*T is needed as a bare minimum

Slide 106 discusses chi-square testing with a focus on the number of parameters.

P factor indicators per time point and T time points result in P*T variables = 6*7 = 42 in our example.

The H0 model may have more parameters than the sample size.

The H1 model has P*T*(P*T+1)/2 = 903 parameters in our example, that is, far more than the sample size of N = 244.

This raises the question of what the quality is of the regular chi-square testing of the H0 model against the unrestricted H1 model.

Simulations based on the estimated model suggest inflated chi-square 5% reject proportions:
- N = 250: 0.41
- N = 500: 0.14
- N = 1000: 0.11

This suggests that for model testing, a sample size of at least 500 should be used for this situation. However, parameter estimates, SEs, and coverage are good even at smaller N.

5.1 EFA longitudinal invariance testing

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - **EFA longitudinal invariance testing**
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- EFA theory

Bengt Muthén Exploratory Factor Analysis 107/ 139

Slide 107 returns to the Outline. We will now turn to invariance testing for longitudinal EFA.

Input for Longitudinal Measurement Invariance Testing

```
              USEVARIABLES = relax1-excit7;
              ! 6 PA items, 3 low arousal, 3 high arousal:
              ! relaxed (pala1) satisfied (pala2) confident (pala3)
              ! happy (paha1) energetic (paha2) excited (paha3)
              ! 7 time points

ANALYSIS:     ESTIMATOR = MLR;
              MODEL = CONFIGURAL METRIC SCALAR;

MODEL:        ! random intercepts for all 6 items:
              i1 BY relax1-relax7@1;
              i2 BY satis1-satis7@1;
              i3 BY conf1-conf7@1;
              i4 BY happy1-happy7@1;
              i5 BY energ1-energ7@1;
              i6 BY excit1-excit7@1;
              ! auto-regressions among factor indicators residuals:
              relax2^-relax7^ PON relax1^-relax6^ (ar1);
              satis2^-satis7^ PON satis1^-satis6^ (ar2);
              conf2^conf7^ PON conf1^-conf6^ (ar3);
              happy2^-happy7^ PON happy1^-happy6^ (ar4);
              energ2^-energ7^ PON energ1^-energ6^ (ar5);
              excit2^-excit7^ PON excit1^-excit6^ (ar6);
```


Slide 108 shows the input for longitudinal measurement invariance testing with EFA. This is using the PA example presented earlier.

The ANALYSIS command uses the MODEL option to test configural, metric, and scalar invariance. The rest of the specification is as shown earlier with random intercepts, and auto-regressions among factor indicator residuals.

The input continues on the next slide.

Input for Longitudinal Measurement Invariance Testing Continued: The Use of Model t

```
            ! AR1 regressions among factors
            ! to reduce the number of parameters:
            f21-f22 ON f11-f12;
            f31-f32 ON f21-f22;
            f41-f42 ON f31-f32;
            f51-f52 ON f41-f42;
            f61-f62 ON f51-f52;
            f71-f72 ON f61-f62;

            i1-i6 with f11-f72@0;

            ! 2-factor ESEM for each time point:
MODEL t1:   f11-f12 by relax1 satis1 conf1 happy1 energ1 excit1(*1);
MODEL t2:   f21-f22 by relax2 satis2 conf2 happy2 energ2 excit2(*2);
MODEL t3:   f31-f32 by relax3 satis3 conf3 happy3 energ3 excit3(*3);
MODEL t4:   f41-f42 by relax4 satis4 conf4 happy4 energ4 excit4(*4);
MODEL t5:   f51-f52 by relax5 satis5 conf5 happy5 energ5 excit5(*5);
MODEL t6:   f61-f62 by relax6 satis6 conf6 happy6 energ6 excit6(*6);
MODEL t7:   f71-f72 by relax7 satis7 conf7 happy7 energ7 excit7(*7);
```


Slide 109 continues the input with the AR1 regressions among the factors.

The bottom part of the slide shows the new feature that can be used with invariance testing. Here, an EFA is set up by ESEM for each timepoint using the MODEL t feature. This creates 7 EFA blocks, one for each timepoint. Invariance testing for these 7 EFA blocks is carried out in combination with using the ANALYSIS option model = configural metric scalar.

Longitudinal Measurement Invariance Test Results

Model	Number of Parameters	χ^2	Degrees of Freedom	P-Value
Configural	212	1246.811	733	0.0000
Metric	164	1281.246	781	0.0000
Scalar	140	1326.299	805	0.0000

Models Compared	χ^2	Degrees of Freedom	P-Value
Metric against Configural	57.341	48	0.1673
Scalar against Configural	94.980	72	0.0362
Scalar against Metric	46.660	24	0.0037

- The scalar model is typically rejected but is needed for growth modeling
- Ways out of this dilemma include:
  - Longitudinal alignment
  - Approximate invariance using PSEM or BSEM, e.g. for only the scalar part (intercepts)

Slide 110 shows the results of the measurement invariance testing. The top part of the table shows that all 3 models are rejected by chi-square. The bottom part suggests that the metric model does not fit significantly worse than the configural model, whereas the scalar models fits borderline worse than the metric model.

The scalar model is often rejected and such rejection is an obstacle to subsequent growth modeling where means are compared over time.

There are ways out of the dilemma of scalar misfit. Alignment is a key technique that will be discussed next. It is also possible to use PSEM, or the Bayesian counterpart BSEM, for approximate scalar invariance with priors for differences in the intercepts.

Automated Invariance Testing Advantages

- Also handles categorical outcomes using WLSMV with Delta and Theta parameterizations
- Automatically uses the scaling correction factors for chi-square difference testing with MLR and uses DIFFTEST with WLSMV

Slide 111 points out advantages of the automated measurement invariance testing using MODEL = configural metric scalar in combination with MODEL t.

One advantage is the handling of categorical outcomes with the WLSMV estimator using both Delta and Theta parameterization. The other advantage is the automation of scaling corrections with chi-square difference testing for MLR and difference testing using DIFFTEST with WLSMV.

Input for Longitudinal Measurement Invariance Testing with Categorical Outcomes: WLSMV

- Three 8-category ordinal items measuring 1 factor at 7 time points

```
          USEVARIABLES = bkThin1f bkThin1s bkThin2s
          bkThin3s bkThin4s bkThin5s bkThin6s
          harmO1f harmO1s harmO2s harmO3s harmO4s harmO5s
          harmO6s
          takeP1f takeP1s takeP2s takeP3s takeP4s TakeP5s takeP6s ;

          CATEGORICAL = bkThin1f - takeP6s;
          MISSING = ALL (999);

ANALYSIS: ESTIMATOR = WLSMV;
          MODEL = CONFIGURAL SCALAR;
```


Slide 112 shows an example of invariance testing with categorical outcomes using the WSLMV estimator. The example has 3 categorical indicators of a factor measured at 7 timepoints. The ANALYSIS command specifies WLSMV estimation and invariance testing.

The input continues on the next slide.

Input Continued

```
MODEL:      ! As before, but optional

MODEL t1:   f1 BY bkthin1f
            harmo1f
            takeP1f ;
MODEL t2:   f2 BY bkthin1s
            harmo1s
            takeP1s ;
MODEL t3:   f3 BY bkthin2s
            harmo2s
            takeP2s ;
MODEL t4:   f4 BY bkthin3s
            harmo3s
            takeP3s ;
MODEL t5:   f5 BY bkthin4s
            harmo4s
            takeP4s ;
MODEL t6:   f6 BY bkthin5s
            harmo5s
            takeP5s;
MODEL t7:   f7 BY bkthin6s
            harmo6s
            takeP6s ;
```


Slide 113 continues the input with the MODEL command which uses the MODEL t approach for the 7 timepoints. Because there is only 1 factor, EFA and ESEM are not relevant and the specification is done in CFA style.

Different Number of Categories for Different Time Points

- Adding one more time point leads to an error:
  - *** ERROR in MODEL command MODEL T1 and MODEL T8 are not equivalent. The categorical indicators in the same position for factors across time must have the same number of categories. Problem with: BKTHIN1F and BKTHIN7S
- With ML and Bayes, this can be handled by the * approach:
  - CATEGORICAL = bkThin1f - takeP7s(*);
- WLSMV cannot handle the * approach
- ML can handle it but requires numerical integration and there are typically too many dimensions of integration due to many factors
- Bayes can handle it and is feasible, but no measurement invariance chi-square testing summary is provided

Slide 114 shows a practical complication when analyzing categorical factor indicators over time, namely having different number of response categories at different timepoints. For instance, some low or high categories may not have observations at the beginning or end of the time series.

The input on the previous slides showed an analysis for the first 7 timepoints. As shown in the first bullet, adding the 8th timepoint leads to an error in that this last timepoint doesn't have the same number of categories as the previous ones.

With ML and Bayes, this can be handled but not by WLSMV.

But ML requires numerical integration of high dimension and is therefore not possible.

Bayes can do it but does not produce invariance testing.

This may need further research.

5.2 EFA longitudinal alignment

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - **EFA longitudinal alignment**
  - EFA growth modeling
- Further topics
- EFA theory

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Slide 115 returns to the Outline, showing that we are still in the longitudinal setting. The next longitudinal topic is alignment.

Longitudinal Alignment

- Section 5.3 of Asparouhov & Muthén (2024). Penalized structural equation models. Structural Equation Modeling: A Multidisciplinary Journal, 31(3), 429–454.
  - 1 factor at several timepoints using the ALIGNMENT option
- PSEM can be used with DIFF priors for factor loadings so that factor variances can be estimated for t>1
  - See the Asparouhov (2023) Mplus Workshop slides 52 - `https://www.statmodel.com/download/M3TeachingSlides.pdf`
- Automated longitudinal alignment for EFA using the ALIGNMENT option

Slide 116 turns to longitudinal alignment. Th alignment approach avoids the problem of ill-fitting metric and scalar invariance over time, relying only on configural invariance.

The slide shows 3 background sources for longitudinal alignment. The 2024 PSEM paper has a section 5.3 giving an example of alignment with 1 factor measured at several timepoints. Clearly, this does not cover the EFA context.

PSEM can also be used with DIFF priors that hold factor loadings approximately equal over time. In this way, factor variance can be estimated for all but the first timepoint. This procedure is described in the 2023 workshop slides by Asparouhov.

Finally, we have the automated approach for EFA that will be described here.

Input for Longitudinal Alignment of PA Factors

```
            USEVARIABLES = relax1-excit7;
            ! 6 PA items, 3 low arousal, 3 high arousal:
            ! relaxed (pala1) satisfied (pala2) confident (pala3)
            ! happy (paha1) energetic (paha2) excited (paha3)

ANALYSIS:   ESTIMATOR = MLR;
            ALIGNMENT = FIXED; ! this is the only change
            ! to the measurement invariance input that used
            ! MODEL = CONFIGURAL etc

MODEL:      ! random intercepts for all 6 items:
            i1 BY relax1-relax7@1;
            i2 BY satis1-satis7@1;
            i3 BY conf1-conf7@1;
            i4 BY happy1-happy7@1;
            i5 BY energ1-energ7@1;
            i6 BY excit1-excit7@1;
            ! auto-regressions among factor indicators residuals:
            relax2^-relax7^ PON relax1^-relax6^ (ar1);
            satis2^-satis7^ PON satis1^-satis6^ (ar2);
            conf2^conf7^ PON conf1^-conf6^ (ar3);
            happy2^-happy7^ PON happy1^-happy6^ (ar4);
            energ2^-energ7^ PON energ1^-energ6^ (ar5);
            excit2^-excit7^ PON excit1^-excit6^ (ar6);
```


Slides 117-118 show the input for the automated alignment approach. As the comment in the ANALYSIS command says, only 1 line needs to be added to the longitudinal invariance input showed earlier, saying ALIGNMENT = FIXED.

The input continues on the next slide.

Input for Longitudinal Alignment Cont'd

```
            ! AR1 regressions among factors
            ! to reduce the number of parameters:
            f21-f22 ON f11-f12;
            f31-f32 ON f21-f22;
            f41-f42 ON f31-f32;
            f51-f52 ON f41-f42;
            f61-f62 ON f51-f52;
            f71-f72 ON f61-f62;

            i1-i6 with f11-f72@0;

            ! 2-factor ESEM for each time point:
MODEL t1:   f11-f12 by relax1 satis1 conf1 happy1 energ1 excit1(*1);
MODEL t2:   f21-f22 by relax2 satis2 conf2 happy2 energ2 excit2(*2);
MODEL t3:   f31-f32 by relax3 satis3 conf3 happy3 energ3 excit3(*3);
MODEL t4:   f41-f42 by relax4 satis4 conf4 happy4 energ4 excit4(*4);
MODEL t5:   f51-f52 by relax5 satis5 conf5 happy5 energ5 excit5(*5);
MODEL t6:   f61-f62 by relax6 satis6 conf6 happy6 energ6 excit6(*6);
MODEL t7:   f71-f72 by relax7 satis7 conf7 happy7 energ7 excit7(*7);
```

Longitudinal Alignment Results: Measurement Part

APPROXIMATE MEASUREMENT INVARIANCE
(NONINVARIANCE) FOR TIMES

Intercepts/Thresholds							
RELAX1	1	2	3	4	5	6	7
SATIS1	1	2	3	4	5	6	7
CONF1	1	2	3	4	5	6	7
HAPPY1	1	2	3	4	5	6	7
ENERG1	1	2	3	4	5	6	7
EXCIT1	1	2	3	4	5	6	7
Loadings for F11							
RELAX1	1	2	3	4	5	6	7
SATIS1	1	2	3	4	5	6	7
CONF1	1	2	3	4	5	6	7
HAPPY1	1	2	3	4	5	6	7
ENERG1	1	2	3	4	5	6	7
EXCIT1	1	2	3	4	5	6	7
Loadings for F12							
RELAX1	1	2	3	4	5	6	7
SATIS1	1	2	3	4	5	6	7
CONF1	1	2	3	4	5	6	7
HAPPY1	1	2	3	4	5	6	7
ENERG1	1	2	3	4	5	6	7
EXCIT1	1	2	3	4	5	6	7

- No parentheses means no measurement noninvariance: All parameters are deemed invariant (scalar invariance)

Slide 119 shows the results of the alignment for the measurement part of the model. The rows represent the measurement parameters. They are intercepts for the 6 indicators and 6 loadings for each of the two EFA factors. The columns represent the 7 timepoints. Entries with parentheses are non-invariant. Because no entry has a parenthesis, all measurement parameters are invariant as judged by alignment.

Longitudinal Alignment Results: Factor Means

FACTOR INTERCEPT COMPARISON AT THE
5% SIGNIFICANCE LEVEL IN DESCENDING ORDER

Results for Factor F11

Factor Ranking	Time	Factor Intercept	Times With Significantly Smaller Factor Intercept
1	5	0.299	1 3 7
2	4	0.002	
3	1	0.000	7
4	2	-0.114	
5	6	-0.137	
6	3	-0.152	
7	7	-0.336	

Results for Factor F12

Factor Ranking	Time	Factor Intercept	Times With Significantly Smaller Factor Intercept
1	5	0.300	1 3 6
2	4	0.236	1 3 6
3	7	0.095	6
4	1	0.000	6
5	2	-0.046	
6	3	-0.055	
7	6	-0.302	

Slide 120 shows the results of the alignment for the factor means. The top part of the table shows the means for the F11 factor. The means are listed from largest to smallest. It is seen that time 5 has the largest mean and that it is significantly larger than the means at times 1, 3, and 7. The factor mean at time 1 is fixed at zero as a result of using the FIXED option.

5.3 EFA growth modeling

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - **EFA growth modeling**
- Further topics
- EFA theory

Slide 121 returns to the Outline of the presentation and shows that we have now reached the final longitudinal EFA topic of growth modeling.

Input for EFA Growth Modeling: PSEM GEOMIN Priors

```
            USEVARIABLES = relax1-excit7;

ANALYSIS:   ESTIMATOR = MLR;
            STARTS = 20;
            ITERATIONS = 2000;

MODEL:      f11-f12 BY relax1*1 satis1 conf1
            happy1 energ1 excit1 (a1-a12);
            f21-f22 BY relax2*1 satis2 conf2
            happy2 energ2 excit2 (a1-a12);
            f31-f32 BY relax3*1 satis3 conf3
            happy3 energ3 excit3 (a1-a12);
            f41-f42 BY relax4*1 satis4 conf4
            happy4 energ4 excit4 (a1-a12);
            f51-f52 BY relax5*1 satis5 conf5
            happy5 energ5 excit5 (a1-a12);
            f61-f62 BY relax6*1 satis6 conf6
            happy6 energ6 excit6 (a1-a12);
            f71-f72 BY relax7*1 satis7 conf7
            happy7 energ7 excit7 (a1-a12);

            f11-f12@1;

            f11 WITH f12;
            f21 WITH f22;
            f31 WITH f32;
            f41 WITH f42;
            f51 WITH f52;
            f61 WITH f62;
            f71 WITH f72;

            i1 s1 q1 c1 | f11@0 f21@.1 f31@.2 f41@.3
            f51@.4 f61@.5 f71@.6;
            i2 s2 q2 c2 | f12@0 f22@.1 f32@.2 f42@.3
            f52@.4 f62@.5 f72@.6;

            q1@0; q2@0; c1@0; c2@0;

            ! scalar invariance for intercepts:
            [relax1-relax7] (int1);
            [satis1-satis7] (int2);
            [conf1-conf7] (int3);
            [happy1-happy7] (int4);
            [energ1-energ7] (int5);
            [excit1-excit7] (int6);

            ! time-invariant residual variances:
            relax1 satis1 conf1 happy1 energ1 excit1 (v1-v6);
            relax2 satis2 conf2 happy2 energ2 excit2 (v1-v6);
            relax3 satis3 conf3 happy3 energ3 excit3 (v1-v6);
            relax4 satis4 conf4 happy4 energ4 excit4 (v1-v6);
            relax5 satis5 conf5 happy5 energ5 excit5 (v1-v6);
            relax6 satis6 conf6 happy6 energ6 excit6 (v1-v6);
            relax7 satis7 conf7 happy7 energ7 excit7 (v1-v6);

MODEL PRIORS:  a1-a12 ~ GEOMIN(2,1,0.0001);

OUTPUT:     STANDARDIZED TECH4;

PLOT:       TYPE = PLOT3;
```

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Slide 122 shows the input for EFA growth modeling of the PA example.

The left column shows the EFA measurement modeling for the 2 factors at each of the 7 timepoints. It is handled by PSEM. The factor loadings are labeled and given GEOMIN priors in the MODEL PRIORS command. Note that the labels are the same at the 7 different timepoints so that metric invariance is imposed. The GEOMIN settings are shown in the EFA Theory section of the presentation.

In the right column, the growth model is specified for each of the two EFA factors using cubic growth.

Scalar invariance is imposed by intercept equalities across time. This is necessary for the growth modeling.

For parsimony, time invariance of the residual variances is also specified.

6 Further topics

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
  - Bi-factor, DSEFA
  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- **Further topics**
- EFA theory

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Slide 123 returns to the Outline of the presentation. We will now turn to Further topics.

Further Topics

- Mixture EFA
  - User's Guide ex 4.4
  - For papers, see the Special Mplus Topic Mixture modeling page: `https://www.statmodel.com/MixtureModeling.shtml`
- Twolevel EFA
  - User's Guide ex 4.5, 4.6
  - Mplus Short Courses Topic 7, Mplus Web Talk 1
- DSEM EFA
  - Asparouhov & Muthén (2026b)

Slide 124 lists some further EFA topics that will not be covered in this presentation, including sources for studies of these topics.

Mixture EFA uses both continuous latent variables and categorical latent variables. EFA is specified for the continuous factors, to which a mixture of 2 or more latent classes is added. Viewed from the perspective of mixture modeling, this allows EFA within-class correlations for latent class/latent profile analysis.

Twolevel EFA makes it possible to do EFA on each of the two levels with different number of factors on the two levels. It is also possible to leave one of the levels unrestricted and only test the number of factors on the other level.

EFA with dynamic structural equation modeling (DSEM) makes is possible to explore the time-invariant measurement structure.

7 EFA theory

Outline

- EFA background
- EFA variations
  - ESEM, PSEM
  - Second-order, SEFA
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  - Target
- EFA in an SEM setting
  - MIMIC
  - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
  - EFA alignment
- EFA in a longitudinal setting
  - EFA longitudinal invariance testing
  - EFA longitudinal alignment
  - EFA growth modeling
- Further topics
- **EFA theory**

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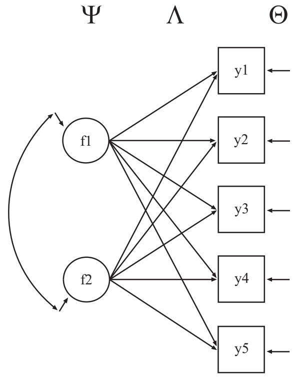
Slide 125 returns to the Outline and it is now time to discuss some more technical matters.

EFA Model with Two Factors (M = 2)



- The goal of EFA is to find the smallest number of factors that explain the correlations among the observed variables
- EFA specifies only the number of factors - unlike confirmatory factor analysis (CFA), EFA has no hypothesis about zero loadings
- The EFA model is not identified but has m^2 indeterminacies
- The EFA model is made identified by applying a "rotation" that eliminates the m^2 indeterminacies and gives an interpretable model

Slide 126 returns to the 2-factor EFA figure shown in the introduction. With this figure in mind, we will now talk a bit more about the issues of indeterminacies and rotation.

EFA Indeterminacies

- Covariance matrix for the p observed variables: $\Sigma = \Lambda\Psi\Lambda^T + \Theta$
  - Λ (lambda) is $p \times m$, Ψ (psi) is $m \times m$, Θ (theta) is $p \times p$
- $\Lambda\Psi\Lambda^T$ has m^2 indeterminacies that need to be handled
  - $m = 1:$ 1 indeterminacy. $\lambda^2\psi = \lambda^{*2}\psi^*$ for $\lambda^* = \lambda/\sqrt{c}$, $\psi^* = \psi c$. Solution: Fix the factor variance ψ at 1.
  - $m = 2$: 4 indeterminacies. Fix the factor variances at 1, factor correlation at 0, and one λ element at 0. This will be called the unrotated model:

$$\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} X & 0 \\ X & X \\ X & X \\ X & X \\ X & X \\ X & X \end{bmatrix} \qquad (1)$$

- The unrotated model usually makes it hard to interpret the factors - rotation is needed

Slide 127 shows the EFA indeterminacies. EFA specifies a model for the covariance matrix of the factor indicators, $\Sigma = \Lambda\Psi\Lambda^T + \Theta$, where Λ is the p*m factor loading matrix, Ψ is the m*m factor covariance matrix, and Θ is a p*p diagonal matrix corresponding to uncorrelated residuals for the p indicators.

As the second bullet points out, the $\Lambda\Psi\Lambda^T$ part of the model has m*m indeterminacies. For m=1, this is seen as two sets of lambda and psi elements giving the same product. We can handle this by fixing the factor variance at 1.

For m=2, the 4 indeterminacies can be handled by fixing the factor variances at 1, factor correlation at 0, and one lambda element at 0. This will be called the unrotated model. An example of this is shown in equation (1). The unrotated model can be seen as a preliminary way to avoid the indeterminacies. It usually makes it hard to interpret the factors. Rotation is needed to get a loading matrix that is simple and thereby easy to interpret.

EFA Rotation

- The indeterminacies can be automatically avoided by applying a simplicity criterion for the model, rotating the unrotated model to a simple factor structure with either orthogonal or oblique factors
- This can be understood in matrix terms, graphical terms, or content terms
- In matrix terms:
  - For $\Psi = I$, the unrotated Λ can be rotated to Λ^* by $\Lambda^* = \Lambda H^{-1}$, where H is an orthogonal matrix, i.e., $H^T = H^{-1}$, which means that $H^{-1}H^{-1T} = I$ so that $\Lambda^*\Lambda^{*T} + \Theta = \Lambda H^{-1}H^{-1T}\Lambda^T + \Theta = \Lambda\Lambda^T + \Theta$
  - This is called an orthogonal rotation, keeping the factors uncorrelated
  - In an oblique rotation, the factors are allowed to be correlated while finding a simple Λ pattern

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Slide 128 shows how the EFA indeterminacy problem is turned into an opportunity to reach a simple, interpretable solution by doing a rotation.

As pointed out in the first bullet, rotation applies a simplicity criterion for the model. Different rotation algorithms used by Mplus are listed on pages 678-682 of the Mplus User's Guide at `https://www.statmodel.com/ugexcerpts.shtml` Theory behind the rotations is discussed in Brown (2001) and more applied accounts are given in Schmitt & Sass (2011) and Morin, Marsh & Nagengast (2013). Many settings can be varied in the rotations for special purposes. For example, drawing on earlier work by Marsh, the Morin et al. paper considers a large Geomin epsilon value of 0.5 for complex loading patterns. For a given data set, it may be useful to try a couple of rotation alternatives to learn more about how the factor structure can be viewed.

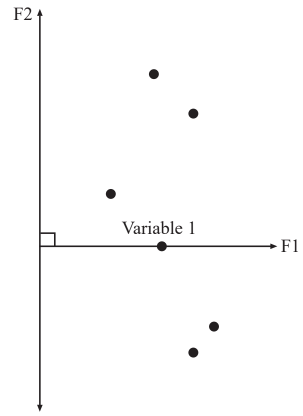
The rotation can be understood in several ways. Let's first consider it in matrix terms and then in graphic terms.

We start with $\Psi = I$, that is, standardized, uncorrelated factors. The unrotated Λ can be rotated to Λ^* by postmultiplying Λ by an orthogonal matrix H where Λ^* is easier to interpret. The formulas on the slide show that this doesn't change the covariance matrix.

So far, this is an orthogonal rotation, keeping the factors uncorrelated. But an oblique rotation can be obtained by allowing the factors to be correlated and thereby obtain a simpler loading pattern.

Matrices → Graph

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \Lambda = \begin{array}{c} \begin{array}{cc} F1 & F2 \end{array} \\ \begin{bmatrix} X & 0 \\ X & X \\ X & X \\ X & X \\ X & X \\ X & X \end{bmatrix} \end{array}$$



- The dots in the figure correspond to the 6 factor indicators
- Correlation between the factors = cosine of the angle
  - 90-degree angle corresponds to correlation = 0
  - 45-degree angle corresponds to correlation = 0.707

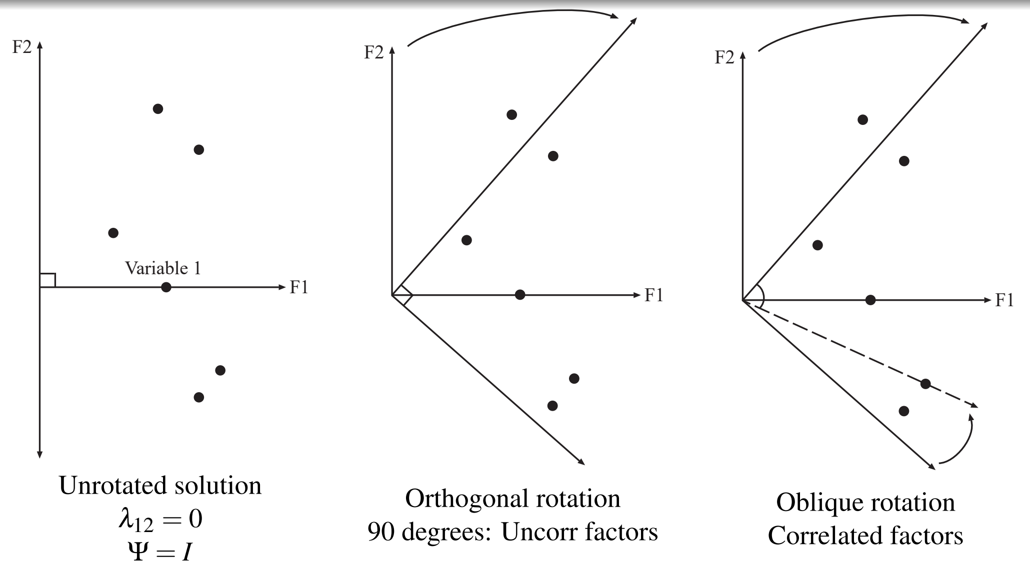
Slide 129 shows the translation of the unrotated factor loading pattern on the left into a 2-dimensional graph on the right.

The 6 dots in the graph represent the 6 factor indicators, that is, the rows of the loading matrix on the left. The x-axis of the graph represents the F1 factor and the y-axis the F2 factor. The zero value for the first indicator's loading on the F2 factor is shown as a dot on the x-axis. In this example, the graph makes up values for the X's in the loading matrix. The first indicator has a zero loading on F2 and a midrange value on F1. Consequently, this is a good indicator of F1, providing an interpretation of the meaning of the F1 factor. The other 5 dots/indicators have substantial loadings on both factors, thereby not contributing to the interpretation of the factors. A rotation is needed to improve interpretability.

As noted at the bottom of the slide, the F1-F2 axes are at a 90-degree angle. This corresponds to a factor correlation of 0, also referred to as orthogonal factors. This is in line with the factor covariance matrix $\Psi = I$.

Note also that a 45-degree angle would represent a correlation of 0.7, obtained as the cosine of the angle.

Factor Loading Rotation: 6 Variables (Dots), 2 Factors



- F1 is interpreted by the 2 variables shown as the 2 lower dots (small loadings on F2)
- F2 is interpreted by the 3 variables shown as the 3 upper dots (small loadings on F1)
- All 3 rotations give the same model fit, just eliminating the indeterminacy differently:
  - The choice of rotation should be based on substantive considerations

Slide 130 shows 3 graphs corresponding to 3 different sets of Ψ and Λ. The left-most graph repeats the representation of the unrotated factor loading matrix shown on the previous slide.

The middle graph shows an orthogonal rotation of the loading matrix. The clock-wise rotation is chosen to improve interpretability by having as many indicators as possible have dots located as close to zero as possible on one of the factor axes. For instance, the top 3 dots have low values on the new, rotated F1 axis. The rotation is orthogonal, that is, it keeps the 90-degree angle between the factors so that we still have $\Psi = I$ as in the unrotated case.

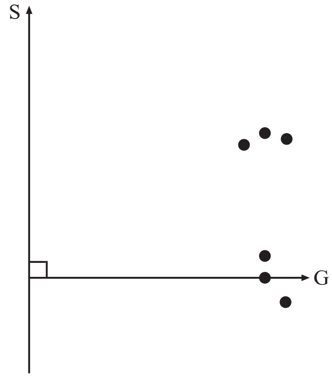
The right-most graph shows an oblique rotation with an angle smaller than 90 so that the factor correlation is greater than 0. It can be seen as the F1 axis moving back up a bit to get the bottom two dots closer to this axis. This oblique rotation therefore further simplifies the interpretation in terms of the loadings.

A summary of the interpretation is given as the bottom of the slide. Note also that all 3 rotations give the same model fit and the choice should be based on substantive considerations.

EFA Rotation: Bifactor vs Correlated Factors

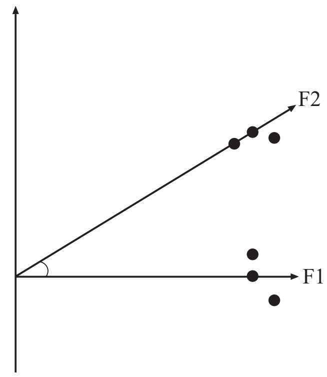
$$\Lambda = \begin{bmatrix} X & 0 \\ X & 0 \\ X & 0 \\ X & X \\ X & X \\ X & X \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\Lambda = \begin{bmatrix} X & 0 \\ X & 0 \\ X & 0 \\ 0 & X \\ 0 & X \\ 0 & X \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1 & X \\ X & 1 \end{bmatrix}$$



- One dataset represented in two different way using two different rotations but having the same model fit

Slide 131 shows a different rotation example. The model on the left and the model on the right have the same fit. It is just the rotation that is different. The 6 dots represent the indicators and their locations are the same for the two models.

The factor loadings on the left can be interpreted as representing a general factor that influences all indicators and one specific factor that influences only the last 3 indicators. The two factors are uncorrelated. On the right, a rotation has been made so that the y-axis has been rotated clock-wise towards the x-axis, resulting in an oblique rotation with a positive factor correlation. The interpretation is that we have two correlated factors, each influencing three of the indicators.

Given that the fit of the two models is the same, this illustrates that substantive considerations determine which rotation is most useful. Assume that the indicators correspond to 6 math tests on arithmetic and calculus. In the right-most model, the factors needed to do well on the tests may be seen as containing not only the specific content of arithmetic and calculus but also a general test taking ability or aptitude for math. This accounts for the factor correlation. In the left-most model, the test taking ability is separated out and the factors are uncorrelated.

Maximum Number of EFA Factors That Can Be Extracted

Number of parameters to be estimated (number of H_0 parameters):

$$a = \underset{\Lambda}{p\,m} + \underset{\Psi}{m\,(m+1)/2} + \underset{\Theta}{p} - m^2$$

where p = number of observed variables, m = number of factors, and m^2 is the number of indeterminacies

b = number of variances/covariances, $p\,(p+1)/2$
(number of H_1 parameters)

Requirement for identification of the EFA model: $a \leq b$. Df = b - a.

Example: $p = 5$ which gives $b = 15$

m = 1: a = 10 df = 5
m = 2: a = 14 df = 1
m = 3: a = 17 nonidentified

Example: $p = 6$ which gives $b = 21$

m = 1: a = 12 df = 9
m = 2: a = 17 df = 4
m = 3: a = 21 df = 0 (just-identified)

Even if $a \leq b$, it may not be possible to extract m factors due to Heywood cases (negative residual variances).

Slide 132 details how number of parameters are decided for EFA and how many factors can be estimated for a given number of indicators.

At the top you see the number of parameters in the 3 parameter arrays, Λ, Ψ, and Θ. From that we subtract the number of indeterminacies m-squared. Let's call the resulting number "a".

Referring to the number of elements in the covariance matrix for the indicators as "b", we note that the model cannot be identified is a is larger than b. The degrees of freedom (df) for chi-square testing is $b - a$.

The two examples show that for 5 indicators, you can extract 2 factors and for 6 indicators you can extract 3 factors.

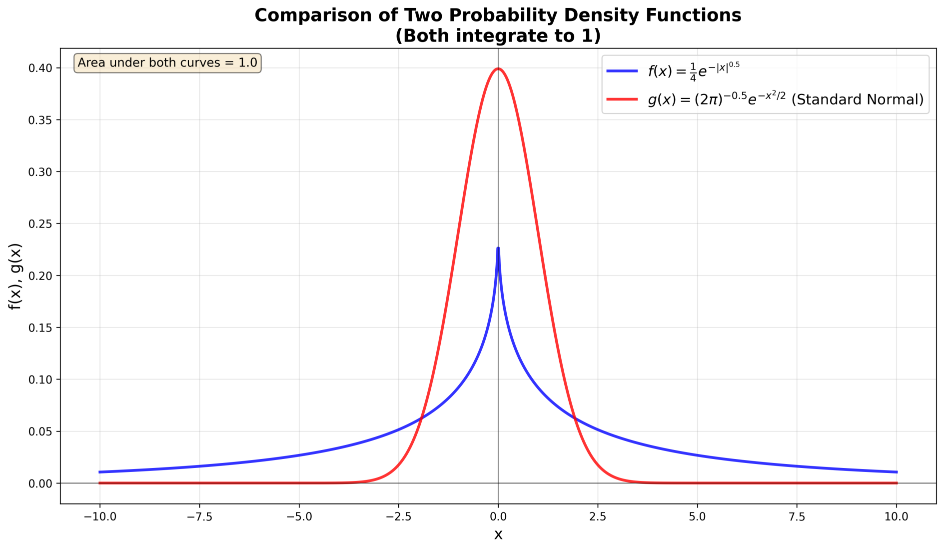
Note also that not all real-data applications result in a successful EFA. A frequent occurrence is obtaining negative residual variances, referred to as Heywood cases. This is not an acceptable model. It is possible to get poor fit for m factors and Heywood cases for m+1 and more factors, implying that a factor model is not suitable for this set of indicators and this dataset.

Settings for GEOMIN, SEFA, and DSEFA Priors

- GEOMIN(m,v,eps) - see Asparouhov & Muthén (2009, 2024)
  - m: number of factors
  - v: prior variance or more specifically 1/(penalty weight)
    - Typically 1.0 or 0.1. No default
  - eps: small positive number controlling penalty smoothness
    - Typically 0.01. Default varies with the number of factors m
- SEFA(v,eps) - see Asparouhov & Muthén (2026)
  - m: number of first-order factors
  - v: GEOMIN prior variance. Default = 0.1
  - eps: GEOMIN eps. Default = 0.01
  - Can be written as: SEFA, SEFA(v), or SEFA(v,eps)
    - The number of first-order factors m of GEOMIN is derived from the input
- DSEFA(v,v2,eps) - see Asparouhov & Muthén (2026)
  - v: GEOMIN prior variance. Default = 0.1
  - v2: ALF(0,v2) prior variance for direct effects. Default = 1
  - eps: GEOMIN eps. Default = 0.01
  - Can be written as: DSEFA, DSEFA(v), DSEFA(v,v2), or DSEFA(v,v2,eps)

Slide 133 shows prior settings for the different rotations achieved by GEOMIN, SEFA, and DSEFA. This slide speaks for itself.

Priors



- Density plots of N(0,1) and ALF(0,1)
- The second argument of ALF is 1/weight of the penalty (the variance is 30/4)

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Slide 134 shows what an ALF prior distribution looks like compared to a normal prior. The blue curve represents ALF.

It is seen that for values close to zero, ALF gives a smaller probability of deviating from zero than the normal distribution. This is important for keeping effects small. For values farther from zero, however, ALF gives a higher probability of deviating from zero.

SEFA - DSEFA Transition

- For SEFA, the first-order factors are captured in the factor loading matrix Λ_1 and the second-order factors are captured in the factor loading matrix Λ_2 with indirect effects from the second-order factors to the indicators computed as the product $\Lambda_1\ \Lambda_2$
- For DSEFA, the general factor loadings are captured in the factor loading matrix $\Lambda_1\ \Lambda_2 + \Lambda_3$ where Λ_3 represents the direct effects from the second-order factor to the indicators
- In DSEFA, Λ_2 is merely a technical construct that should not be substantively interpreted - however:
- It is helpful to consider the transition from SEFA to DSEFA:
  - When $\Lambda_3 = 0$, SEFA is at hand and Λ_2 is central to interpretation
  - With a few significant Λ_3 loadings, we are rejecting the notion of only indirect effects from the second-order factor to the indicators so that a second-order factor is no longer sufficient
  - With several significant Λ_3 loadings, we have clearly transitioned to DSEFA and we then leave Λ_2's interpretation behind, instead focusing on the factor loading matrix $\Lambda_1\ \Lambda_2 + \Lambda_3$

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Slide 135 discusses the factor loading matrices of SEFA and DSEFA. The corresponding formulas are given in Asparouhov & Muthén (2026), equations (8)–(18).

For SEFA, the first-order factors are captured in the factor loading matrix Λ_1 and the second-order factors are captured in the factor loading matrix Λ_2, with indirect effects from the second-order factors to the indicators computed as the product $\Lambda_1\Lambda_2$.

For DSEFA, the general factor loadings are captured in the factor loading matrix $\Lambda_1\Lambda_2 + \Lambda_3$, where Λ_3 represents the direct effects from the second-order factor to the indicators. In DSEFA, Λ_2 is merely a technical construct that should not be substantively interpreted — however, it is helpful in considering the transition from SEFA to DSEFA as follows:
- When $\Lambda_3 = 0$, SEFA is at hand and Λ_2 is central to interpretation
- With a few significant Λ_3 loadings, we are rejecting the notion of only indirect effects from the second-order factor to the indicators so that a second-order factor is no longer sufficient
- With several significant Λ_3 loadings, we have clearly transitioned to DSEFA and we then leave Λ_2's interpretation behind, instead focusing on the factor loading matrix $\Lambda_1\Lambda_2 + \Lambda_3$.

The remaining slides provide references.

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