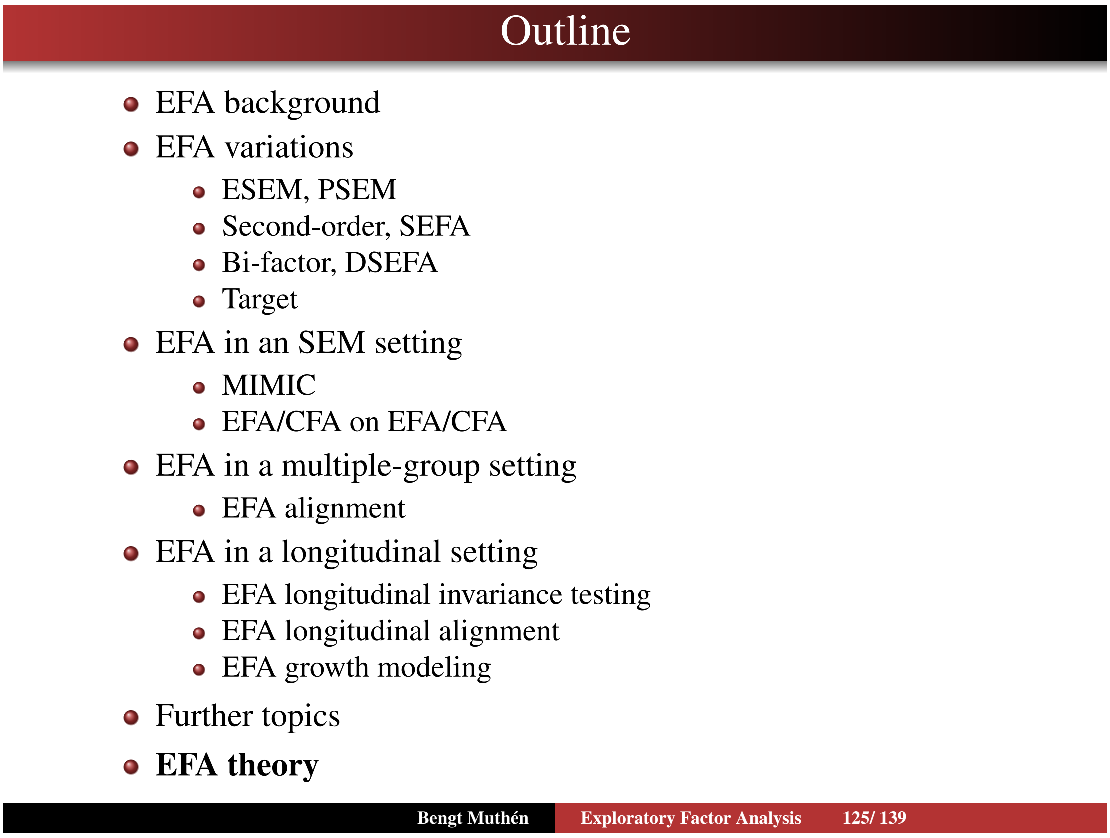
7 EFA theory

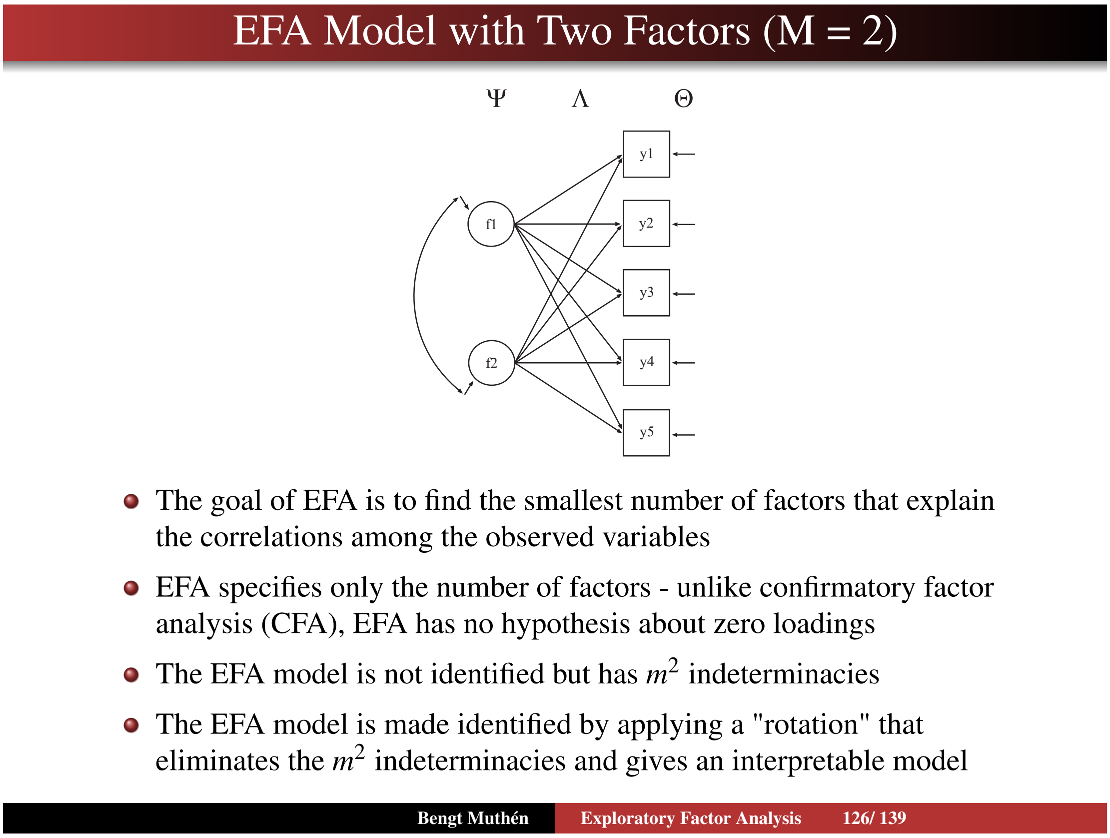


Outline

- EFA background
- EFA variations
    - ESEM, PSEM
    - Second-order, SEFA
    - Bi-factor, DSEFA
    - Target
- EFA in an SEM setting
    - MIMIC
    - EFA/CFA on EFA/CFA
- EFA in a multiple-group setting
    - EFA alignment
- EFA in a longitudinal setting
    - EFA longitudinal invariance testing
    - EFA longitudinal alignment
    - EFA growth modeling
- Further topics
- **EFA theory**

Bengt Muthén Exploratory Factor Analysis 125/ 139

Slide 125 returns to the Outline and it is now time to discuss some more technical matters.

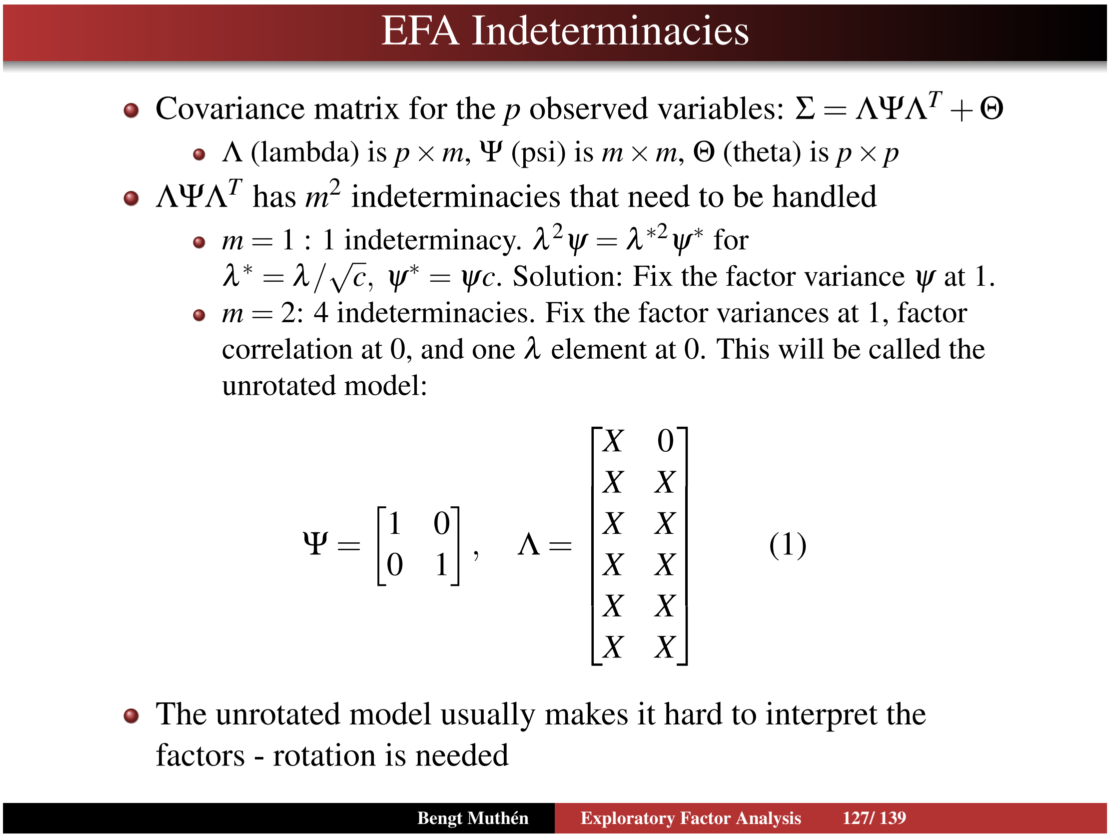


EFA Model with Two Factors (M = 2)

- The goal of EFA is to find the smallest number of factors that explain the correlations among the observed variables
- EFA specifies only the number of factors - unlike confirmatory factor analysis (CFA), EFA has no hypothesis about zero loadings
- The EFA model is not identified but has m^2 indeterminacies
- The EFA model is made identified by applying a "rotation" that eliminates the m^2 indeterminacies and gives an interpretable model

Bengt Muthén **Exploratory Factor Analysis** **126/ 139**

Slide 126 returns to the 2-factor EFA figure shown in the introduction. With this figure in mind, we will now talk a bit more about the issues of indeterminacies and rotation.



EFA Indeterminacies

- Covariance matrix for the p observed variables: $\Sigma = \Lambda \Psi \Lambda^T + \Theta$
  - Λ (lambda) is $p \times m$, Ψ (psi) is $m \times m$, Θ (theta) is $p \times p$
- $\Lambda \Psi \Lambda^T$ has m^2 indeterminacies that need to be handled
  - $m = 1 : 1$ indeterminacy. $\lambda^2 \psi = \lambda^{*2} \psi^*$ for $\lambda^* = \lambda / \sqrt{c}$, $\psi^* = \psi c$. Solution: Fix the factor variance ψ at 1.
  - $m = 2$: 4 indeterminacies. Fix the factor variances at 1, factor correlation at 0, and one λ element at 0. This will be called the unrotated model:

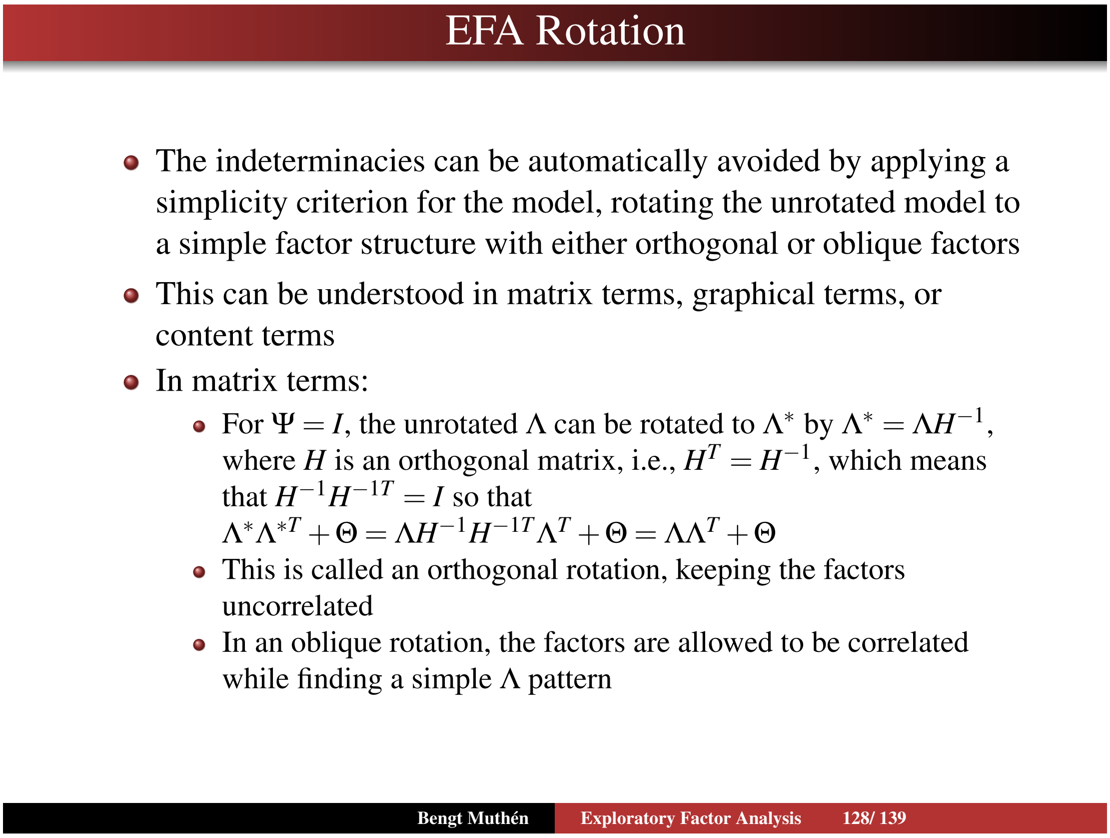
$$\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} X & 0 \\ X & X \\ X & X \\ X & X \\ X & X \\ X & X \end{bmatrix} \qquad (1)$$

- The unrotated model usually makes it hard to interpret the factors - rotation is needed

Slide 127 shows the EFA indeterminacies. EFA specifies a model for the covariance matrix of the factor indicators, $\Sigma = \Lambda \Psi \Lambda^T + \Theta$, where Λ is the p*m factor loading matrix, Ψ is the m*m factor covariance matrix, and Θ is a p*p diagonal matrix corresponding to uncorrelated residuals for the p indicators.

As the second bullet points out, the $\Lambda \Psi \Lambda^T$ part of the model has m*m indeterminacies. For m=1, this is seen as two sets of lambda and psi elements giving the same product. We can handle this by fixing the factor variance at 1.

For m=2, the 4 indeterminacies can be handled by fixing the factor variances at 1, factor correlation at 0, and one lambda element at 0. This will be called the unrotated model. An example of this is shown in equation (1). The unrotated model can be seen as a preliminary way to avoid the indeterminacies. It usually makes it hard to interpret the factors. Rotation is needed to get a loading matrix that is simple and thereby easy to interpret.



EFA Rotation

- The indeterminacies can be automatically avoided by applying a simplicity criterion for the model, rotating the unrotated model to a simple factor structure with either orthogonal or oblique factors
- This can be understood in matrix terms, graphical terms, or content terms
- In matrix terms:
    - For $\Psi = I$, the unrotated Λ can be rotated to Λ^* by $\Lambda^* = \Lambda H^{-1}$, where H is an orthogonal matrix, i.e., $H^T = H^{-1}$, which means that $H^{-1} H^{-1T} = I$ so that $\Lambda^* \Lambda^{*T} + \Theta = \Lambda H^{-1} H^{-1T} \Lambda^T + \Theta = \Lambda \Lambda^T + \Theta$
    - This is called an orthogonal rotation, keeping the factors uncorrelated
    - In an oblique rotation, the factors are allowed to be correlated while finding a simple Λ pattern

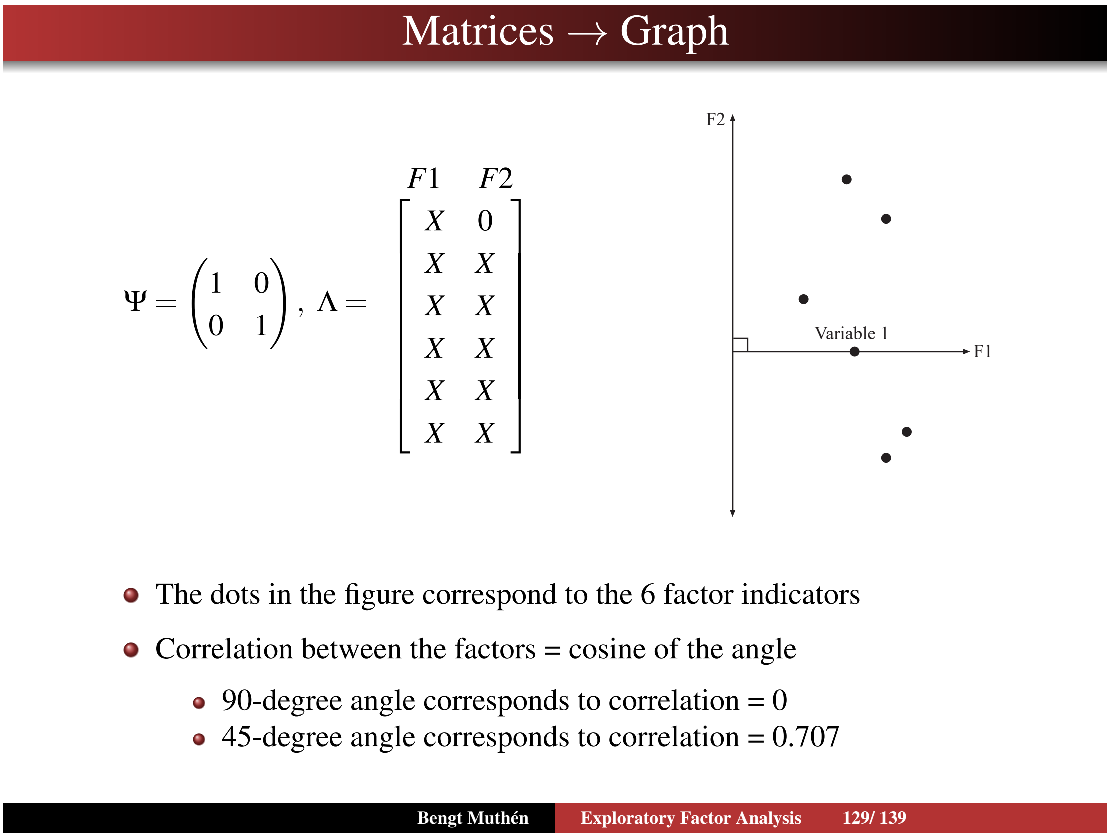
Slide 128 shows how the EFA indeterminacy problem is turned into an opportunity to reach a simple, interpretable solution by doing a rotation.

As pointed out in the first bullet, rotation applies a simplicity criterion for the model. Different rotation algorithms used by Mplus are listed on pages 678-682 of the Mplus User's Guide at `https://www.statmodel.com/ugexcerpts.shtml` Theory behind the rotations is discussed in Brown (2001) and more applied accounts are given in Schmitt & Sass (2011) and Morin, Marsh & Nagengast (2013). Many settings can be varied in the rotations for special purposes. For example, drawing on earlier work by Marsh, the Morin et al. paper considers a large Geomin epsilon value of 0.5 for complex loading patterns. For a given data set, it may be useful to try a couple of rotation alternatives to learn more about how the factor structure can be viewed.

The rotation can be understood in several ways. Let's first consider it in matrix terms and then in graphic terms.

We start with $\Psi = I$, that is, standardized, uncorrelated factors. The unrotated Λ can be rotated to Λ^* by postmultiplying Λ by an orthogonal matrix H where Λ^* is easier to interpret. The formulas on the slide show that this doesn't change the covariance matrix.

So far, this is an orthogonal rotation, keeping the factors uncorrelated. But an oblique rotation can be obtained by allowing the factors to be correlated and thereby obtain a simpler loading pattern.



Matrices → Graph

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \Lambda = \begin{array}{c} \begin{array}{cc} F1 & F2 \end{array} \\ \begin{bmatrix} X & 0 \\ X & X \\ X & X \\ X & X \\ X & X \\ X & X \end{bmatrix} \end{array}$$

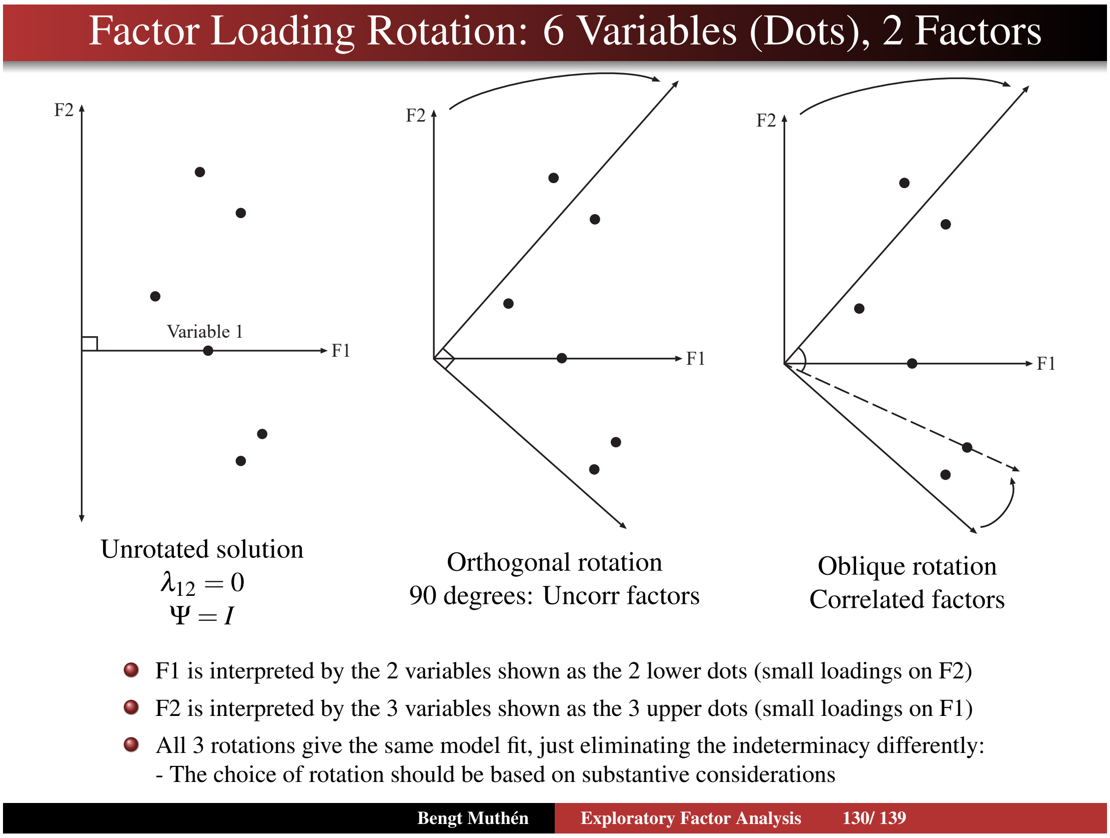
- The dots in the figure correspond to the 6 factor indicators
- Correlation between the factors = cosine of the angle
    - 90-degree angle corresponds to correlation = 0
    - 45-degree angle corresponds to correlation = 0.707

Slide 129 shows the translation of the unrotated factor loading pattern on the left into a 2-dimensional graph on the right.

The 6 dots in the graph represent the 6 factor indicators, that is, the rows of the loading matrix on the left. The x-axis of the graph represents the F1 factor and the y-axis the F2 factor. The zero value for the first indicator's loading on the F2 factor is shown as a dot on the x-axis. In this example, the graph makes up values for the X's in the loading matrix. The first indicator has a zero loading on F2 and a midrange value on F1. Consequently, this is a good indicator of F1, providing an interpretation of the meaning of the F1 factor. The other 5 dots/indicators have substantial loadings on both factors, thereby not contributing to the interpretation of the factors. A rotation is needed to improve interpretability.

As noted at the bottom of the slide, the F1-F2 axes are at a 90-degree angle. This corresponds to a factor correlation of 0, also referred to as orthogonal factors. This is in line with the factor covariance matrix $\Psi = I$.

Note also that a 45-degree angle would represent a correlation of 0.7, obtained as the cosine of the angle.



Factor Loading Rotation: 6 Variables (Dots), 2 Factors

Unrotated solution
$\lambda_{12} = 0$
$\Psi = I$

Orthogonal rotation
90 degrees: Uncorr factors

Oblique rotation
Correlated factors

- F1 is interpreted by the 2 variables shown as the 2 lower dots (small loadings on F2)
- F2 is interpreted by the 3 variables shown as the 3 upper dots (small loadings on F1)
- All 3 rotations give the same model fit, just eliminating the indeterminacy differently:
  - The choice of rotation should be based on substantive considerations

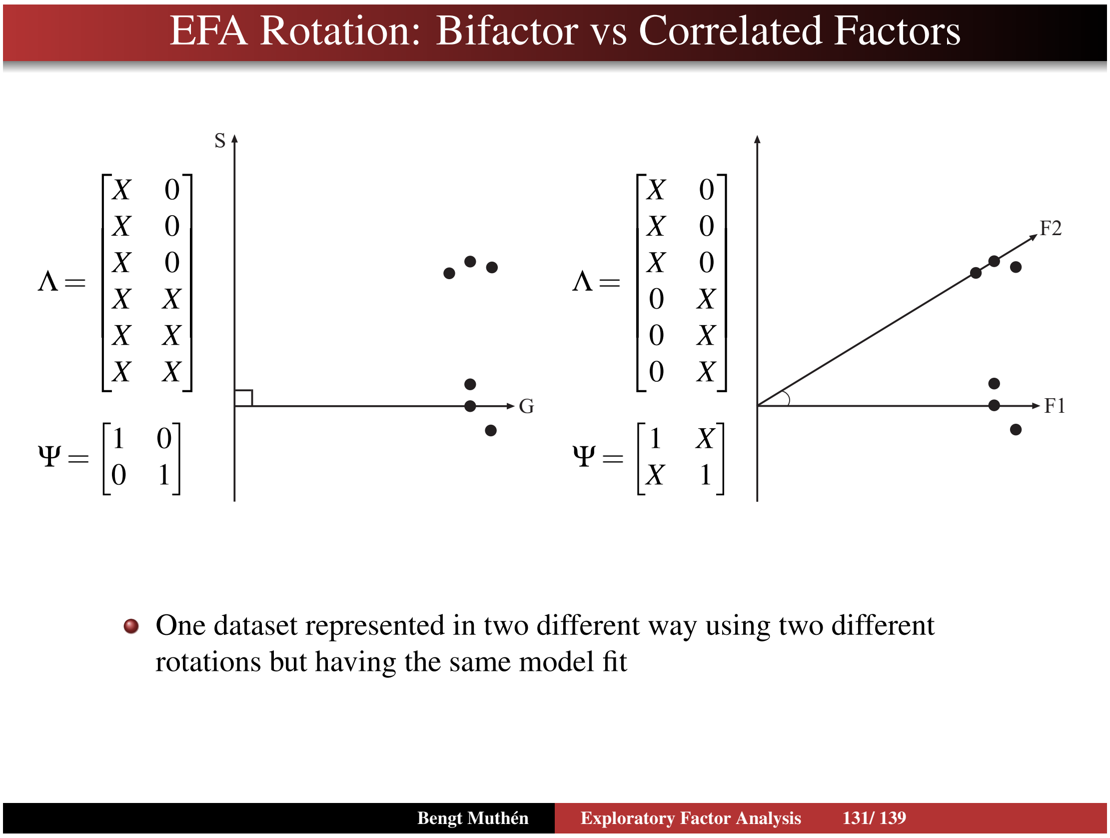
Bengt Muthén **Exploratory Factor Analysis** **130/ 139**

Slide 130 shows 3 graphs corresponding to 3 different sets of Ψ and Λ. The left-most graph repeats the representation of the unrotated factor loading matrix shown on the previous slide.

The middle graph shows an orthogonal rotation of the loading matrix. The clock-wise rotation is chosen to improve interpretability by having as many indicators as possible have dots located as close to zero as possible on one of the factor axes. For instance, the top 3 dots have low values on the new, rotated F1 axis. The rotation is orthogonal, that is, it keeps the 90-degree angle between the factors so that we still have $\Psi = I$ as in the unrotated case.

The right-most graph shows an oblique rotation with an angle smaller than 90 so that the factor correlation is greater than 0. It can be seen as the F1 axis moving back up a bit to get the bottom two dots closer to this axis. This oblique rotation therefore further simplifies the interpretation in terms of the loadings.

A summary of the interpretation is given as the bottom of the slide. Note also that all 3 rotations give the same model fit and the choice should be based on substantive considerations.



EFA Rotation: Bifactor vs Correlated Factors

$$\Lambda = \begin{bmatrix} X & 0 \\ X & 0 \\ X & 0 \\ X & X \\ X & X \\ X & X \end{bmatrix} \qquad \Lambda = \begin{bmatrix} X & 0 \\ X & 0 \\ X & 0 \\ 0 & X \\ 0 & X \\ 0 & X \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Psi = \begin{bmatrix} 1 & X \\ X & 1 \end{bmatrix}$$

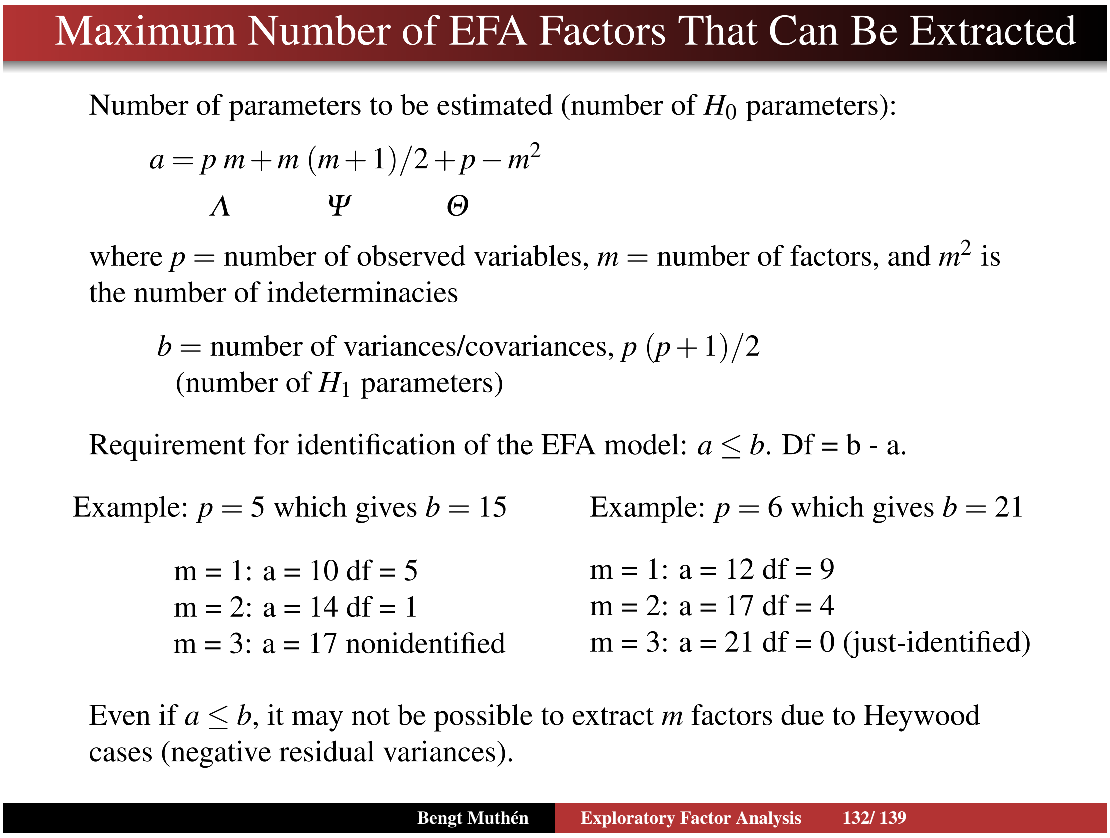
- One dataset represented in two different way using two different rotations but having the same model fit

Bengt Muthén Exploratory Factor Analysis 131/ 139

Slide 131 shows a different rotation example. The model on the left and the model on the right have the same fit. It is just the rotation that is different. The 6 dots represent the indicators and their locations are the same for the two models.

The factor loadings on the left can be interpreted as representing a general factor that influences all indicators and one specific factor that influences only the last 3 indicators. The two factors are uncorrelated. On the right, a rotation has been made so that the y-axis has been rotated clock-wise towards the x-axis, resulting in an oblique rotation with a positive factor correlation. The interpretation is that we have two correlated factors, each influencing three of the indicators.

Given that the fit of the two models is the same, this illustrates that substantive considerations determine which rotation is most useful. Assume that the indicators correspond to 6 math tests on arithmetic and calculus. In the right-most model, the factors needed to do well on the tests may be seen as containing not only the specific content of arithmetic and calculus but also a general test taking ability or aptitude for math. This accounts for the factor correlation. In the left-most model, the test taking ability is separated out and the factors are uncorrelated.



Maximum Number of EFA Factors That Can Be Extracted

Number of parameters to be estimated (number of H_0 parameters):

$$a = \underset{\Lambda}{p\,m} + \underset{\Psi}{m\,(m+1)/2} + \underset{\Theta}{p} - m^2$$

where $p =$ number of observed variables, $m =$ number of factors, and m^2 is the number of indeterminacies

$b =$ number of variances/covariances, $p\,(p+1)/2$
(number of H_1 parameters)

Requirement for identification of the EFA model: $a \leq b$. Df = b - a.

Example: $p = 5$ which gives $b = 15$	Example: $p = 6$ which gives $b = 21$
m = 1: a = 10 df = 5	m = 1: a = 12 df = 9
m = 2: a = 14 df = 1	m = 2: a = 17 df = 4
m = 3: a = 17 nonidentified	m = 3: a = 21 df = 0 (just-identified)

Even if $a \leq b$, it may not be possible to extract m factors due to Heywood cases (negative residual variances).

Slide 132 details how number of parameters are decided for EFA and how many factors can be estimated for a given number of indicators.

At the top you see the number of parameters in the 3 parameter arrays, Λ, Ψ, and Θ. From that we subtract the number of indeterminacies m-squared. Let's call the resulting number "a".

Referring to the number of elements in the covariance matrix for the indicators as "b", we note that the model cannot be identified is a is larger than b. The degrees of freedom (df) for chi-square testing is $b - a$.

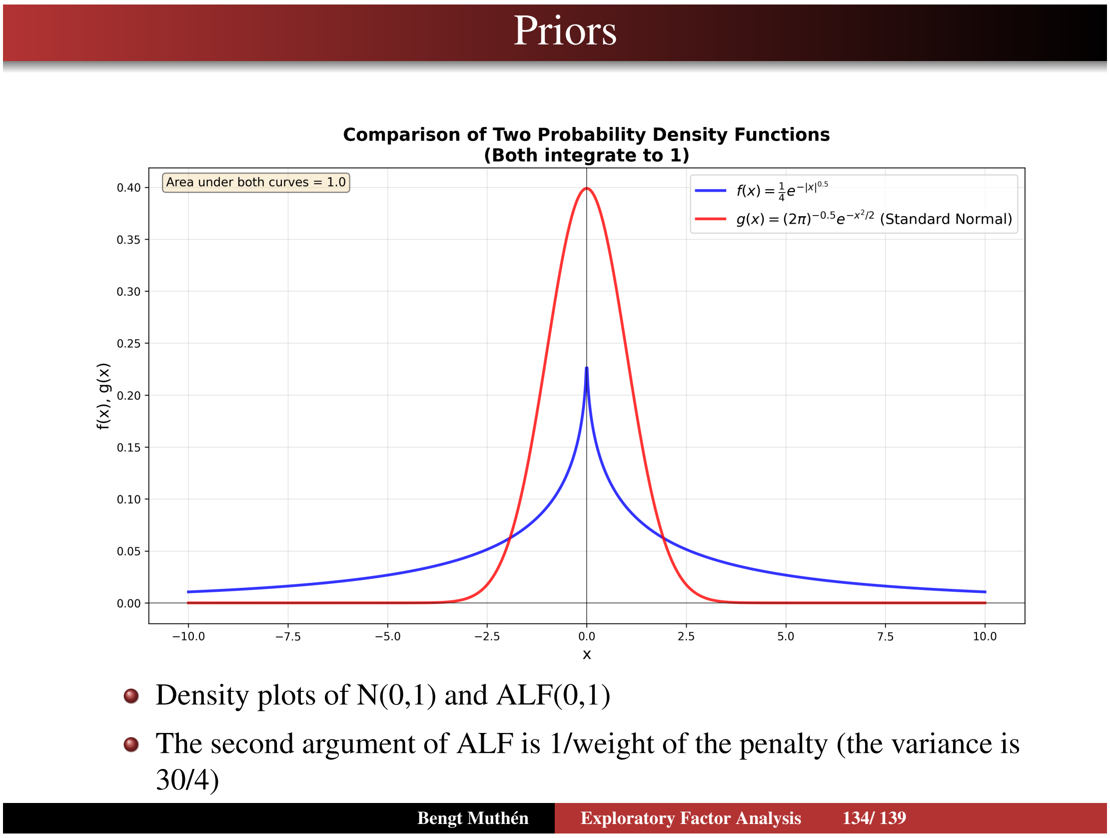
The two examples show that for 5 indicators, you can extract 2 factors and for 6 indicators you can extract 3 factors.

Note also that not all real-data applications result in a successful EFA. A frequent occurrence is obtaining negative residual variances, referred to as Heywood cases. This is not an acceptable model. It is possible to get poor fit for m factors and Heywood cases for m+1 and more factors, implying that a factor model is not suitable for this set of indicators and this dataset.

Settings for GEOMIN, SEFA, and DSEFA Priors

- GEOMIN(m,v,eps) - see Asparouhov & Muthén (2009, 2024)
    - m: number of factors
    - v: prior variance or more specifically 1/(penalty weight)
      - Typically 1.0 or 0.1. No default
    - eps: small positive number controlling penalty smoothness
      - Typically 0.01. Default varies with the number of factors m
- SEFA(v,eps) - see Asparouhov & Muthén (2026)
    - m: number of first-order factors
    - v: GEOMIN prior variance. Default = 0.1
    - eps: GEOMIN eps. Default = 0.01
    - Can be written as: SEFA, SEFA(v), or SEFA(v,eps)
      - The number of first-order factors m of GEOMIN is derived from the input
- DSEFA(v,v2,eps) - see Asparouhov & Muthén (2026)
    - v: GEOMIN prior variance. Default = 0.1
    - v2: ALF(0,v2) prior variance for direct effects. Default = 1
    - eps: GEOMIN eps. Default = 0.01
    - Can be written as: DSEFA, DSEFA(v), DSEFA(v,v2), or DSEFA(v,v2,eps)

Slide 133 shows prior settings for the different rotations achieved by GEOMIN, SEFA, and DSEFA. This slide speaks for itself.

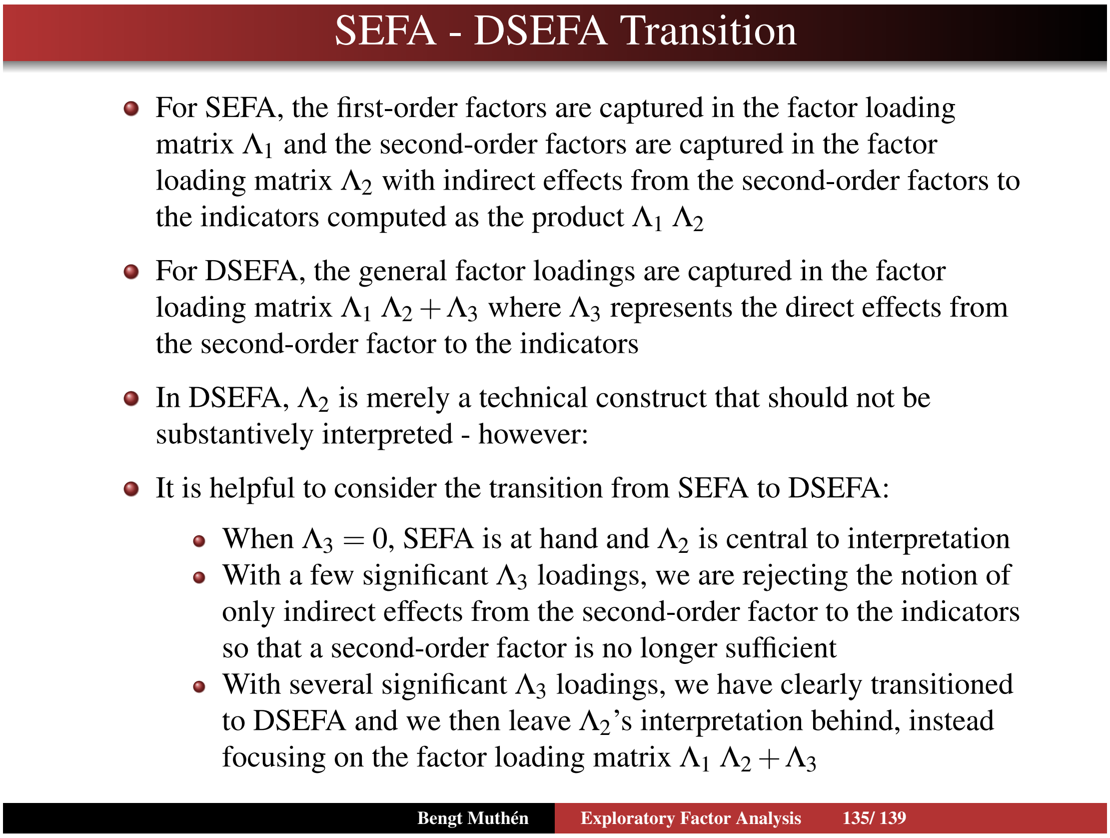


Priors

- Density plots of N(0,1) and ALF(0,1)
- The second argument of ALF is 1/weight of the penalty (the variance is 30/4)

Slide 134 shows what an ALF prior distribution looks like compared to a normal prior. The blue curve represents ALF.

It is seen that for values close to zero, ALF gives a smaller probability of deviating from zero than the normal distribution. This is important for keeping effects small. For values farther from zero, however, ALF gives a higher probability of deviating from zero.



SEFA - DSEFA Transition

- For SEFA, the first-order factors are captured in the factor loading matrix Λ_1 and the second-order factors are captured in the factor loading matrix Λ_2 with indirect effects from the second-order factors to the indicators computed as the product $\Lambda_1\ \Lambda_2$
- For DSEFA, the general factor loadings are captured in the factor loading matrix $\Lambda_1\ \Lambda_2 + \Lambda_3$ where Λ_3 represents the direct effects from the second-order factor to the indicators
- In DSEFA, Λ_2 is merely a technical construct that should not be substantively interpreted - however:
- It is helpful to consider the transition from SEFA to DSEFA:
  - When $\Lambda_3 = 0$, SEFA is at hand and Λ_2 is central to interpretation
  - With a few significant Λ_3 loadings, we are rejecting the notion of only indirect effects from the second-order factor to the indicators so that a second-order factor is no longer sufficient
  - With several significant Λ_3 loadings, we have clearly transitioned to DSEFA and we then leave Λ_2's interpretation behind, instead focusing on the factor loading matrix $\Lambda_1\ \Lambda_2 + \Lambda_3$

Bengt Muthén Exploratory Factor Analysis 135/ 139

Slide 135 discusses the factor loading matrices of SEFA and DSEFA. The corresponding formulas are given in Asparouhov & Muthén (2026), equations (8)–(18).

For SEFA, the first-order factors are captured in the factor loading matrix Λ_1 and the second-order factors are captured in the factor loading matrix Λ_2, with indirect effects from the second-order factors to the indicators computed as the product $\Lambda_1\Lambda_2$.

For DSEFA, the general factor loadings are captured in the factor loading matrix $\Lambda_1\Lambda_2 + \Lambda_3$, where Λ_3 represents the direct effects from the second-order factor to the indicators. In DSEFA, Λ_2 is merely a technical construct that should not be substantively interpreted — however, it is helpful in considering the transition from SEFA to DSEFA as follows:
- When $\Lambda_3 = 0$, SEFA is at hand and Λ_2 is central to interpretation
- With a few significant Λ_3 loadings, we are rejecting the notion of only indirect effects from the second-order factor to the indicators so that a second-order factor is no longer sufficient
- With several significant Λ_3 loadings, we have clearly transitioned to DSEFA and we then leave Λ_2's interpretation behind, instead focusing on the factor loading matrix $\Lambda_1\Lambda_2 + \Lambda_3$.

The remaining slides provide references.