

Bayesian Analysis In Mplus:
A Brief Introduction

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Incomplete Draft, Version 2

May 10, 2010

Abstract

This paper uses a series of examples to give an introduction to how Bayesian analysis is carried out in Mplus. The examples are a mediation model with estimation of an indirect effect, a structural equation model, a two-level regression model with estimation of a random intercept variance, a multiple-indicator binary growth model with a large number of latent variables, a two-part growth model, and a mixture model. It is shown how the use of Mplus graphics provides information on estimates, convergence, and model fit. Comparisons are made with frequentist estimation using maximum likelihood and weighted least squares. Data and Mplus scripts are available on the Mplus website.

1 Introduction

Frequentist (e.g., maximum likelihood) and Bayesian analysis differ by the former viewing parameters as constants and the latter as variables. Maximum likelihood (ML) finds estimates by maximizing a likelihood computed for the data. Bayes combines prior distributions for parameters with the data likelihood to form posterior distributions for the parameter estimates. The priors can be diffuse (non-informative) or informative where the information may come from previous studies. The posterior provides an estimate in the form of a mean, median, or mode of the posterior distribution.

There are many books on Bayesian analysis and most are quite technical. Gelman et al. (2004) provides a good general statistical description, whereas Lynch (2010) gives a somewhat more introductory account. Lee (2007) gives a discussion from a structural equation modeling perspective. Schafer (1997) gives a statistical discussion from a missing data and multiple imputation perspective, whereas Enders (2010) gives an applied discussion of these same topics. Statistical overview articles include Gelfand et al. (1990) and Casella and George (1992). Overview articles of an applied nature and with a latent variable focus include Scheines et al. (1999), Rupp et al. (2004), and Yuan and MacKinnon (2009).

Bayesian analysis is firmly established in mainstream statistics. Its popularity is growing and currently appears to be featured at least half as often as frequentist analysis. Part of the reason for the increased use of Bayesian analysis is the success of new computational algorithms referred to as Markov chain Monte Carlo (MCMC) methods. Outside of statistics, however, application of Bayesian analysis lags behind. One possible reason is that Bayesian analysis is perceived as difficult to do, requiring complex statistical specifications such as those used in the flexible, but technically-oriented general Bayes program WinBUGS. These observations were the background for developing Bayesian analysis in Mplus (Muthén & Muthén, 1998-2010). In Mplus, simple analysis specifications

with convenient defaults allow easy access to a rich set of analysis possibilities. Diffuse priors are used as the default with the possibility of specifying informative priors. A range of graphics options are available to easily provide information on estimates, convergence, and model fit.

Three key points motivate taking an interest in Bayesian analysis:

1. More can be learned about parameter estimates and model fit
2. Analyses can be made less computationally demanding
3. New types of models can be analyzed

Point 1 is illustrated by parameter estimates that do not have a normal distribution. ML gives a parameter estimate and its standard error and assumes that the distribution of the parameter estimate is normal based on asymptotic (large-sample) theory. In contrast, Bayes does not rely on large-sample theory and provides the whole distribution not assuming that it is normal. The ML confidence interval $Estimate \pm 1.96 \times SE$ assumes a symmetric distribution, whereas the Bayesian credibility interval based on the percentiles of the posterior allows for a strongly skewed distribution. Bayesian exploration of model fit can be done in a flexible way using Posterior predictive checking (PPC; see, e.g., Gelman et al., 1996; Gelman et al., 2004, Chapter 6; Lee, 2007, Chapter 5; Scheines et al., 1999). Any suitable test statistics for the observed data can be compared to statistics based on simulated data obtained via draws of parameter values from the posterior distribution, avoiding statistical assumptions about the distribution of the test statistics. Examples of non-normal posteriors are presented in Section 2 for single-level models as well as in Section 4 for multilevel models. Examples of PPC are given in Section 3.

Point 2 may be of interest for an analyst who is hesitant to move from ML estimation to Bayesian estimation. Many models are computationally cumbersome or impossible using ML, such as with categorical outcomes and many latent variables resulting in many dimensions of numerical integration. Such an analyst may view the Bayesian analysis

simply as a computational tool for getting estimates that are analogous to what would have been obtained by ML had it been feasible. This is obtained with diffuse priors, in which case ML and Bayesian results are expected to be close in large samples (Browne & Draper, 2006; p. 505). Examples of this are presented in Section 5.

Point 3 is exemplified by models with a very large number of parameters or where ML does not provide a natural approach. Examples of the former include image analysis (see, e.g., Green, 1996)) and examples of the latter include random change-point analysis (see, e.g., Dominicus et al., 2008).

This paper gives a brief introduction to Bayesian analysis as implemented in Mplus. For a technical discussion of this implementation, see Asparouhov and Muthén (2010). Section 2 provides two mediation modeling examples which illustrate a non-normal posterior, how to use priors, and how to do a basic Bayes analysis in Mplus. Section 3 uses a CFA example to illustrate both informative priors and the use of PPC. Section 4 uses two-level regression to illustrate how to get correct a correct assessment of the size of a skewed random effect variance estimate and intraclass correlation. Section 5 uses multiple-indicator growth for binary items to illustrate high-dimensional analysis. Section 6 uses a two-part growth model to illustrate Bayes speed advantage over ML with many dimensions. Section 7 uses a mixture model to illustrate how to handle label switching. Section 8 discusses alternative approaches to missing data modeling. Data and Mplus scripts are available on the Mplus web site under Mplus Examples, Applications using Mplus.

2 Two mediation modeling examples

Two mediation modeling examples are considered. The first example uses the ATLAS data of MacKinnon et al. (2004) and illustrates how different conclusions about the intervention effect are arrived at using ML versus Bayes. The second example uses the firefighter data

of Yuan and MacKinnon (2009) to illustrate the use of priors based on information from previous studies to shorten the credibility interval (the Bayesian counterpart to confidence intervals) for the intervention effect.

2.1 The ATLAS example: Different conclusions using Bayes vs ML

The mediational model in Figure 1 was considered in MacKinnon et al. (2004). The intervention program ATLAS (Adolescent Training and Learning to Avoid Steroids) was administered to high school football players to prevent use of anabolic steroids. MacKinnon et al. (2004) used a sample of $n = 861$ with complete data from 15 treatment schools and 16 control schools (the multilevel nature of the data was ignored and is ignored here as well; multilevel Bayesian mediational modeling is, however, available in Mplus). One part of the intervention aimed at increasing perceived severity of using steroids. This in turn was hypothesized to increase good nutrition behaviors. In Figure 1 these three variables are denoted tx , $steroid$, and $nutrition$, respectively.

[Figure 1 about here.]

A key parameter is the indirect effect of intervention on the nutrition outcome, $a \times b$. The ML point estimate (SE) for this is 0.020 (0.011) with an asymptotically-normal z test value of 1.913. Because this z value is not greater than 1.96, the indirect effect of the intervention is not deemed significant at the 5% level. Correspondingly, the ML 95% confidence interval obtained by the CINTERVAL option of the OUTPUT command is $0 - 0.041$, that is, not excluding zero.

The Mplus input for the corresponding Bayesian analysis is shown in Table 1. The only change is to replace ESTIMATOR = ML with ESTIMATOR = BAYES. When two processors are used faster computations are obtained with the default of two MCMC chains

to be discussed below. Note that the indirect effect is defined as the NEW parameter "indirect" in MODEL CONSTRAINT. Bayesian graphics are obtained with the option TYPE = PLOT2.

[Table 1 about here.]

The analysis results are shown in Table 2. The first column gives the point estimate, which by default is the median of the posterior distribution. The mean or mode can be obtained using the POINT option of the ANALYSIS command. The second column gives the standard deviation of the posterior distribution. A normally distributed z ratio is not used in Bayesian analysis. The third column gives a one-tailed p-value based on the posterior distribution. For a positive estimate, the p-value is the proportion of the posterior distribution that is below zero. For a negative estimate, the p-value is the proportion of the posterior distribution that is above zero. The fourth and fifth columns give the 2.5 and 97.5 percentiles in the posterior distribution, resulting in a 95% Bayesian credibility interval.

Using the default posterior median point estimate, the indirect effect estimate is 0.016, that is, slightly lower than the ML value with a slightly higher posterior distribution standard deviation of 0.013. Unlike the ML confidence interval, the Bayesian 95% credibility interval of 0.002 – 0.052 does not include zero, implying a positive intervention effect. The reason for this ML-Bayes discrepancy is found when studying the posterior distribution of the indirect effect.

[Table 2 about here.]

View Graphs is used to open the Bayes graphs. The first menu item is Bayesian posterior parameter distributions. Staying in histogram mode and selecting Parameter 8, indirect shows a skewed distribution for the indirect effect with a median of 0.01644; see Figure 2. A smoother picture of the posterior distribution is obtained by requesting the

Kernel density (Botev et al., 2010) option instead of histogram; see Figure 3. The mean, median, and mode are marked as vertical lines in the distribution and are different due to the skewness of the distribution. It is clear that the ML normality assumption is not suitable for the indirect effect parameter (for technical arguments, see also MacKinnon, 2008). Therefore, the symmetric confidence interval that ML uses is not appropriate. Instead, the Bayesian credibility interval uses the 2.5 and 97.5 percentiles of the posterior distribution, allowing for skewness.

[Figure 2 about here.]

It is important to carefully consider convergence in Bayesian analysis. The default convergence criterion is that a Proportional Scale Reduction (PSR) factor is close enough to 1 for each parameter. For a technical definition, see Gelman and Rubin (1992), Gelman et al. (2004), and Asparouhov and Muthén (2010). Briefly stated, Bayesian analysis uses Markov chain Monte Carlo (MCMC) algorithms to iteratively obtain an approximation to the posterior distributions of the parameters from which the estimates are obtained as means, medians, or modes. Such iterations are referred to as a chain. In Mplus several such chains are carried out in parallel when using multiple processors. The PSR approach to determining convergence compares the parameter variation within each chain to that across chains to make sure that the different chains do not converge to different values. The PSR criterion essentially requires the between-chain variation to be small relative to the total of between- and within-chain variation. Mplus uses the default of two chains which usually gives good PSR information that compares well with using more chains. The first half of the iterations are considered as a "burn-in" phase and are not used to represent the posterior distribution.

[Figure 3 about here.]

Figure 4 shows a trace plot of the two chains for the residual variance of the *severity* variable. It is seen how a starting value is improved in a few iterations. A total of 100 iterations are used at which point the convergence criterion is fulfilled. The TECH8 output shown on the screen and printed at the end of the output shows that for this example the largest PSR value at 100 iterations is 1.037. In the trace plot the burn-in phase is denoted by a vertical line at 50 iterations. The last 50 iterations appear to show a stable process with no upward or downward trend with the two chains overlapping in their variation. The posterior distribution shown in Figure 2 is determined by the remaining 50 values for each of the two chains, resulting in a distribution based on 100 points.

[Figure 4 about here.]

A good approach to gain evidence of convergence is to run longer chains and check that the parameter values have not changed in important ways and that the PSR still remains close to 1. The Mplus option FBITERATIONS in the ANALYSIS command can be used to request a fixed number of Bayes iterations. Requesting 10,000 iterations gives the progression of PSR values in TECH8 seen in Table 3. Only the first 1000 iterations are shown.

When FBITERATIONS is used the PSR convergence criterion is not applied by Mplus. Although Mplus prints THE MODEL ESTIMATION TERMINATED NORMALLY convergence has to be verified by the user when using FBITERATIONS. The results are shown in Table 4. The estimates are rather close to those shown earlier for 100 iterations in Table 2.

[Table 3 about here.]

[Table 4 about here.]

Figure 5 shows the posterior distribution for the indirect effect. It is now smoother than in Figure 2 due to using longer chains, but still shows the skewness.

[Figure 5 about here.]

A further check of the posterior distributions is obtained by an autocorrelation plot. Figure 6 shows this for the indirect effect in the 10,000 iteration analysis. Autocorrelations show the degree of correlatedness of parameter values across iterations for different lags (intervals in the chain). A small value is desirable to obtain approximately independent draws from the posterior. A value of 0.1 or lower has been suggested. If the autocorrelation is high for small lags but decreases with increasing lags, using only every k th iteration can be accomplished by thinning using the THIN option of the ANALYSIS command. Thinning is also useful when a large number of iterations is needed for convergence, but a smaller number is desired for displaying the posterior distributions to reduce computer storage (this influences the size of the Mplus .gph file).

[Figure 6 about here.]

2.2 The firefighter example: Using informative priors

Yuan and MacKinnon (2004) discusses the benefits of Bayesian analysis for mediational analysis. Here it is shown how their firefighter example is carried out in Mplus. The example uses x to represent exposure to the randomized experiment, m to represent change in knowledge of the benefits of eating fruit and vegetables, and y to represent reported eating of fruits and vegetables. The model diagram is analogous to Figure 1. The sample size is $n = 354$. The focus is on the indirect effect $a \times b$. This example illustrates the use of priors. As a first step the analysis is done using the default of diffuse priors. The Mplus input is shown in Table 5 and the output in Table 6. Yuan and MacKinnon (2009) give the corresponding WinBUGS code in their appendix. Table 6 shows a positive indirect effect with 95% credibility interval 0.011 – 0.117.

The firefighter example is used to illustrate the benefit of informative priors. Yuan and MacKinnon (2009, p. 311) put priors on the a and b slopes based on previous studies to

show how this can reduce the width of the credibility interval for the indirect effect of the intervention. A normal prior is used for both slopes. For a the prior has mean 0.35 and variance 0.04 and for b the prior has mean 0.1 and variance 0.01. The prior variances are four times larger than what had been observed in previous studies in order to take into account possible differences in the current study. The Mplus input is shown in Table 7. The MODEL PRIORS command uses the labels a and b to apply the normal priors.

[Table 5 about here.]

[Table 6 about here.]

[Table 7 about here.]

The output given in Table 8 shows a 95% credibility interval of 0.012–0.102. Compared to the Table 6 analysis with the default diffuse priors, this represents a 16% shortening of the credibility interval. The results agree with those of Yuan and MacKinnon (2009). With a smaller sample, the prior has a stronger effect. Figure 7 shows that the indirect effect has a skewed posterior distribution.

[Table 8 about here.]

[Figure 7 about here.]

2.3 Mediation modeling summary

The mediation modeling showed that the indirect effect has a non-normal distribution which is allowed for in the posterior distribution of Bayesian analysis. This can result in incorrect conclusions when using regular ML due to the ML assumption of symmetric, normal distributions for parameter estimates. Bayesian analysis also allows the incorporation of prior information about parameter values, resulting in shorter intervals

for the intervention effect. The use of informative priors clearly needs to be approached with caution. An investigator must not choose a certain prior because it makes it more likely to find an intervention effect. Priors need to draw on information from other similar studies.

An alternative approach to the Bayesian analysis discussed here is to use ML estimates combined with bootstrap-generated confidence intervals that do not assume a symmetric, normal distribution (see, e.g., MacKinnon et al., 1994; Shrout & Bolger, 2002). Such procedures are available in Mplus as described in Muthén & Muthén (1998-2010). An interesting line of research is to study the relative performance of Bayes versus ML with bootstrap.

3 Bayes CFA: Using informative priors for small cross-loadings instead of fixing them to zero

The context of confirmatory factor analysis (CFA) is used to introduce a new Bayesian concept, Posterior Predictive Checking (PPC). With continuous outcomes, PPC in Mplus builds on the standard likelihood-ratio chi-square statistic in mean- and covariance-structure modeling, where a specific H_0 model is tested against the unrestricted H_1 model. This PPC procedure is described in Scheines et al. (1999) and Asparouhov and Muthén (2010), drawing on Gelman et al. (1996). This statistic is evaluated for the data at the Bayes-estimated parameter values. It is compared to a distribution of such values obtained by many replications of the sequence: (1) drawing Bayes parameter estimates from the posterior distribution; (2) drawing a sample of synthetic observations on the outcomes given the parameter values obtained in (1); and (3) computing the likelihood-ratio chi-square statistic for this synthetic sample. The extremeness of the value of the observed-data statistic is evaluated using the synthetic-data statistics by where it falls in

the synthetic-data test statistic distribution. An extreme value shows that the model does not adequately represent the data.

This section also demonstrates the use of the Deviance Information Criterion (DIC; Spiegelhalter et al., 2002; Gelman et al., 2004). Models with small DIC values are preferred. DIC is a Bayesian generalization of the AIC and BIC which draw on ML estimation and balance the largeness of the likelihood with a penalty for the number of parameters expended. For DIC the number of parameters used in the penalty is the effective number of parameters referred to as p_D .

3.1 Holzinger-Swineford mental abilities study

As an example of a confirmatory factor analysis (CFA) using Bayesian estimation, data are considered from the classic 1939 study by Holzinger and Swineford (1939). Twenty-six tests intended to measure a general factor and five specific factors were administered to seventh and eighth grade students in two schools, the Grant-White School ($n = 145$) and the Pasteur School ($n = 156$). Students from the Grant-White School came from homes where the parents were American-born and this sample is considered here.

Factor analyses of these data have been described e.g. by Harman (1976; pp. 123-132). Of the 26 tests, nineteen were intended to measure different specific domains, five were intended to measure general deduction and two were revisions/new test versions. Typically, the last two are not analyzed. Excluding the five general deduction tests, 19 tests measuring four domains are considered here: spatial ability, verbal ability, speed, and recognition/memory. The design of the measurement of the four domains by the 19 tests is shown in the factor loading pattern matrix of Table 9. Here, an X denotes a free loading to be estimated and 0 a fixed, zero loading. This corresponds to a simple structure CFA model with variable complexity one, that is, each variable loads on only one factor.

[Table 9 about here.]

3.2 Simple structure CFA

Table 10 shows the Mplus input for a Bayesian CFA with a simple structure where an item loads on only its hypothesized factor. To obtain convergence, ML starting values and 20,000 iterations in each of the two default chains are requested in the ANALYSIS command. The factor metric is determined by fixing the factor variances at one.

[Table 10 about here.]

Table 11 shows that the model is rejected based on the Bayesian Posterior Predictive Checking (PPC) using the likelihood-ratio chi-square statistic (Asparouhov & Muthén, 2010). The Posterior Predictive P-value (PPP) is 0.006. The 95% confidence interval for the difference between the observed-data test statistic and the replicated-data test statistic is 17 to 112, indicating that the observed-data statistic is much larger than what would have been generated by the model. The corresponding PPC distribution is shown in Figure 8. The location of the observed-data test statistic is marked by a vertical line at zero. The corresponding PPC scatter plot is shown in Figure 9, where the proportion of points above the 45 degree line corresponds to the p-value. These type of displays are further discussed in Gelman et al. (2004, p. 164). The model rejection is in line with ML likelihood-ratio chi-square testing which also rejects the model and obtains a p-value of 0.0002 (CFI = 0.93, RMSEA = 0.057, SRMR = 0.063).

[Table 11 about here.]

[Figure 8 about here.]

[Figure 9 about here.]

Despite the poor fit, it is of interest to display the estimates from the simple structure CFA as a basis for comparison with revised models. Table 12 gives the parameter estimates,

except for the intercepts and the residual variances. The factor loadings are presented column-wise in the factor loading matrix, one factor at a time.

[Table 12 about here.]

3.3 Adding priors for cross-loadings

The simple structure CFA hypothesizes that an item loads on only one factor, that is, no items have cross-loadings. This can be argued to be a strongly simplified representation of the real measurement situation and not essential to the notion that each item is designed to measure a specific factor. A more realistic hypothesis might be that each item has a major loading on the hypothesized factor but that small cross-loadings are possible due to a minor influence on the item from some of the other factors. For example a spatial item may involve a speed or memory component due to how the tests were formulated. To reflect this, cross-loadings can be given informative priors where the mean is zero, and where the variance is small but not zero. The specification of this type of Bayesian CFA is shown in Table 13, where a normal prior with a small variance is specified for each cross-loading. These cross-loadings are not all identified in the sense that if an ML analysis freed all of them the model would not be identified. The loadings are in a standardized metric because the items are standardized and the factor variances are fixed at one. This is a scale-free model so the model fit, standardized solution, and confidence/credibility intervals are the same for items that have been standardized or not (the item standardization is seen in Table 10). A variance prior of 0.01 is chosen. It implies that the range of cross-loading values two standard deviations away from the prior mean of zero is -0.2 to 0.2 , that is, approximately 95% of the prior is within this rather narrow range. Several other variances for the priors were explored where the 0.01 choice resulted in the smallest Deviance Information Criterion (DIC). The variance (DIC) values were: 0.001 (6986), 0.01 (6967), 0.05 (6971), 0.1 (6975), 0.5 (no convergence).

[Table 13 about here.]

Table 14 shows that the Bayes model with priors for cross-loadings matches the data rather well with a PPP of 0.353. The corresponding distribution of test statistics based on data generated from the parameter posteriors and the position of the real-data test statistic are shown in Figure 10. The vertical line is now more centrally located in the distribution on the left and a substantial portion of the points in the scatter plot on the right are above the 45-degree line. This is in contrast to the poor match between the model and data shown in Figure 8 and Figure 9 for the original CFA. The DIC value has also improved from the original model's 6997 to the current model's 6967.

[Table 14 about here.]

[Figure 10 about here.]

All cross-loadings are small. The spatial factor loadings are shown in Table 15, where cross-loadings are displayed between the red horizontal lines. Only the three spatial factor items addition, straight, and object have one-tailed p-values less than 0.05. The three items are also the ones that have the largest modification indices in the ML analysis of the simple structure CFA. It is interesting to see that Bayesian analysis can in this way provide a counterpart to the use of ML modification indices for modifying a model. Table 16 shows that the factor correlations are smaller in this analysis than in the simple structure CFA of Table 12. This is a useful side effect of letting the correlations between the items be channeled through more factors instead of only their primary factors which might force inflated factor correlations.

[Table 15 about here.]

[Table 16 about here.]

3.4 Freeing cross-loadings

As a final step in the Bayesian CFA, the three cross-loadings for the spatial factor, addition, straight and object, are freed. Table 17 shows the input and Table 18 - Table 20 give output excerpts. Only the loadings for the spatial factor are shown. The Posterior Predictive p-value is 0.173. The corresponding p-value for the likelihood-ratio chi-square test in the ML analysis with these three free cross-loadings is 0.0492. The lower p-value for ML may indicate that the LRT is somewhat more powerful than the PPC. The model may have further cross-loadings that should be free, but this is not investigated here. The DIC value has now improved to 6958 from the value 6967 of the earlier model with informative priors.

Table 19 shows that the spatial factor loadings that were freed for the three items addition, straight, and object are larger than in Table 15. This is because default diffuse priors have replaced the informative small-variance priors for these three loadings.

[Table 17 about here.]

[Table 18 about here.]

[Table 19 about here.]

[Table 20 about here.]

An alternative Bayesian analysis adds to the model with three free cross-loadings by using informative priors for the remaining cross-loadings with the small prior variance applied earlier. This may indicate that new cross-loadings should be freed. A summary of DIC and PPC model test results for this model and the previous three is shown in Table 21, also adding the ML results.

[Table 21 about here.]

3.5 Bayes CFA summary

The Bayes CFA example showed another use of informative priors. A considerably better fit to the data was obtained by avoiding the ML use of cross-loadings fixed exactly at zero and instead allowing cross-loadings to have priors with a narrow range around zero. The example showed that model fit can be assessed by PPC and models can be compared using DIC. Model modification was possible based on posterior estimates of cross-loadings. The idea of using priors with small variances for cross-loadings can be generalized to other model settings. An example is MIMIC modeling where all direct effects from covariates to factor indicators cannot be identified but can be given such priors and thereby point to model modification in terms of freeing direct effects.

An alternative approach that also avoids fixing cross-loadings to zero is exploratory factor analysis (EFA), which has been generalized to exploratory structural equation modeling (ESEM; Asparouhov & Muthén, 2009). A four-factor EFA gives the ML p-value of 0.248. Compared to model 3 in Table 21 this suggests that more cross-loadings should be freed. EFA, however, may not be optimal in cases with a strong hypothesis and well-known measurements in that it moves further away from the original simple structure CFA hypothesis and only maintains the hypothesis of the number of factors. Bayes stays with the original CFA model while allowing minor cross-loadings.

PPC procedures including the likelihood-ratio chi-square used here need more research to gauge their performance in terms of power to detect different kinds of misspecification at different sample sizes. Alternative PPC test statistics can be explored. It should be noted that PPC p-values were obtained also for the just-identified mediation models of Section 2. In such cases the testing is mostly in terms of normality assumptions being fulfilled or not. Sensitivity of PPC to non-normality needs to be explored.

4 Conclusions

Further sections will follow in the next version of this paper. Comments are invited on the discussion so far, including aspects that could be further clarified or added. Please send your email to bmuthen@ucla.edu.

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Figure 1: Mediation model for the ATLAS example

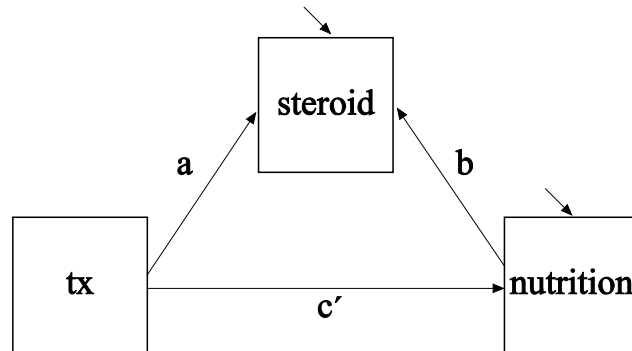


Figure 2: Indirect effect posterior distribution for the ATLAS example

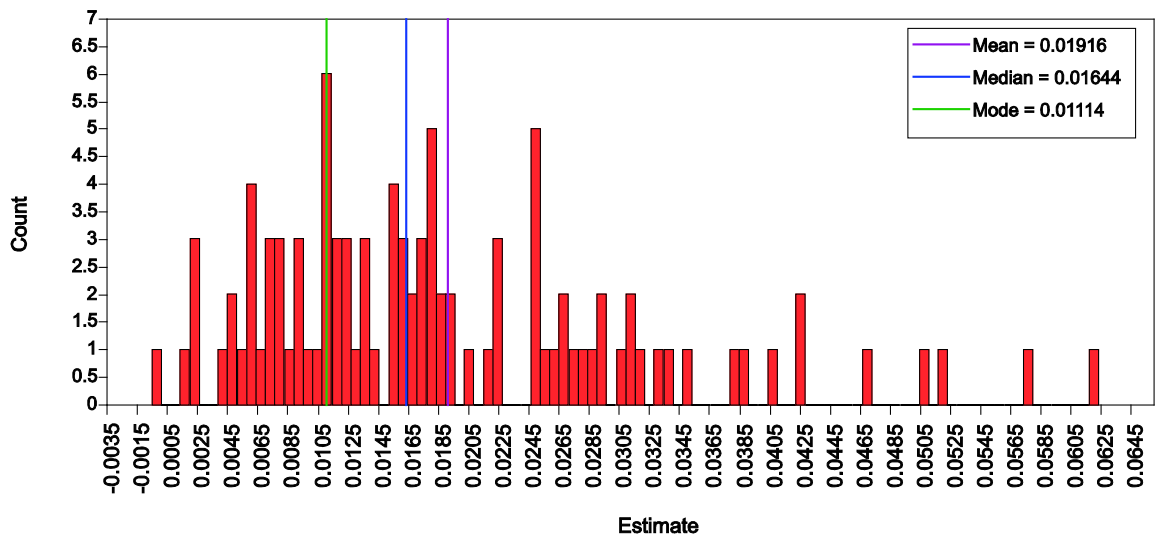


Figure 3: Kernel density estimate of indirect effect posterior distribution for the ATLAS example

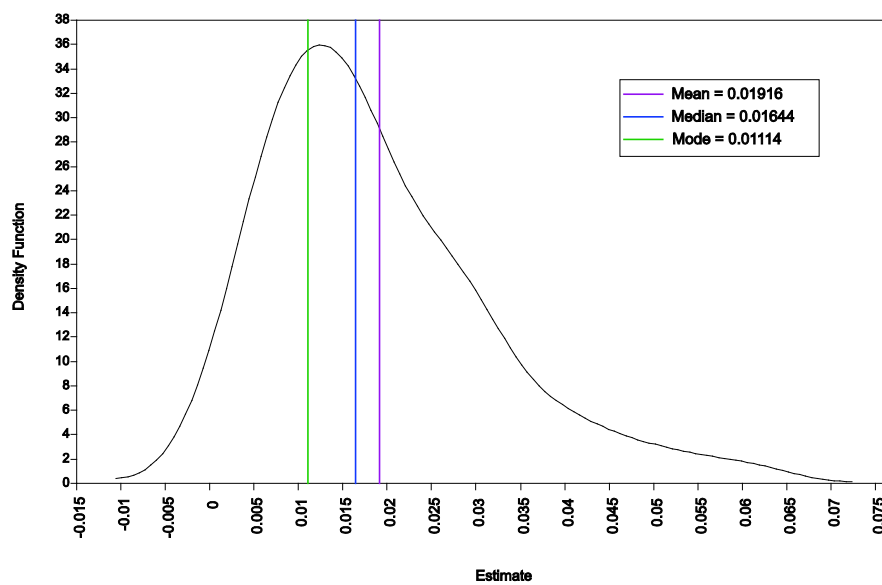


Figure 4: Trace plot of severity residual variance for the ATLAS example

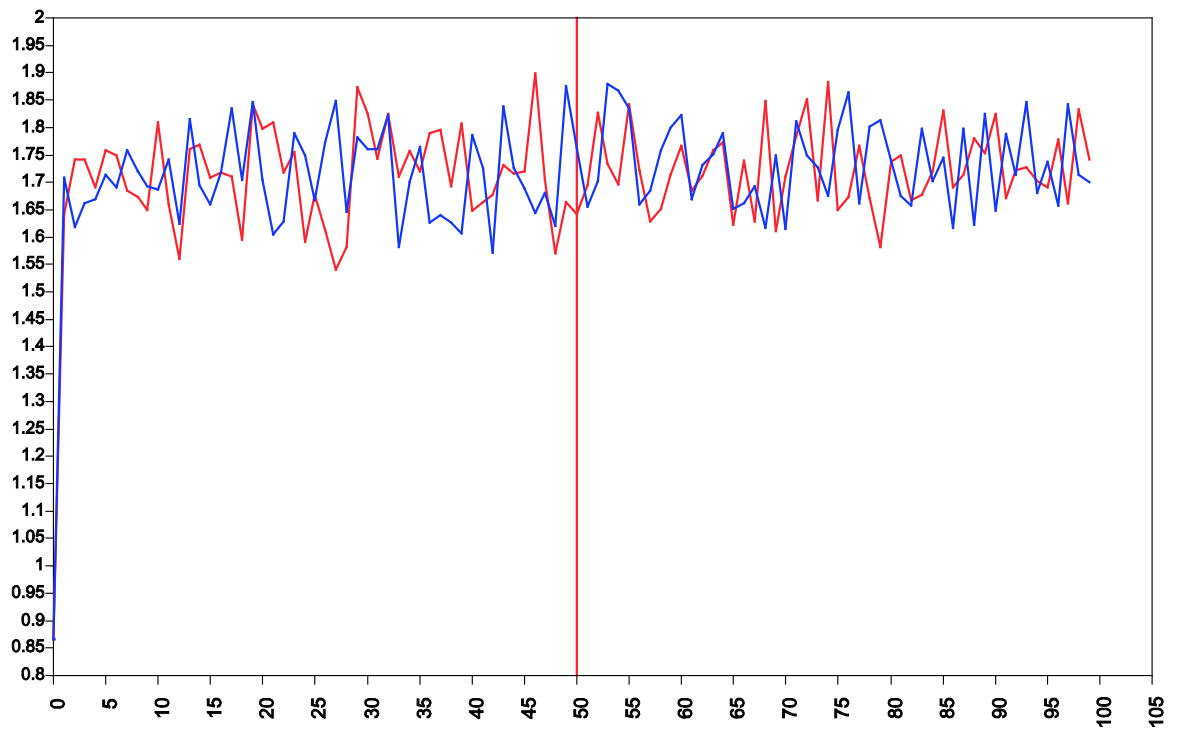


Figure 5: Indirect effect posterior distribution from 10,000 iterations for the ATLAS example

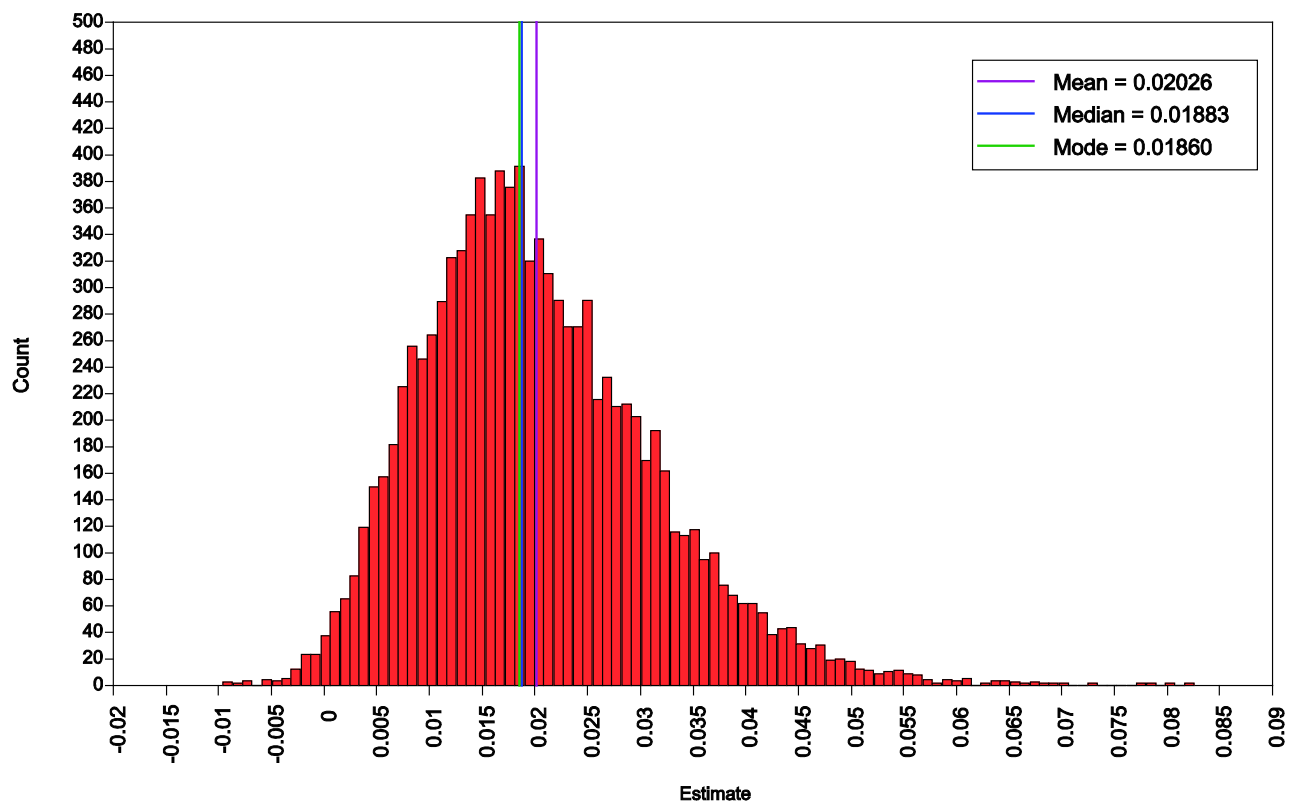


Figure 6: Autocorrelation plot for indirect effect from 10,000 iterations for the ATLAS example

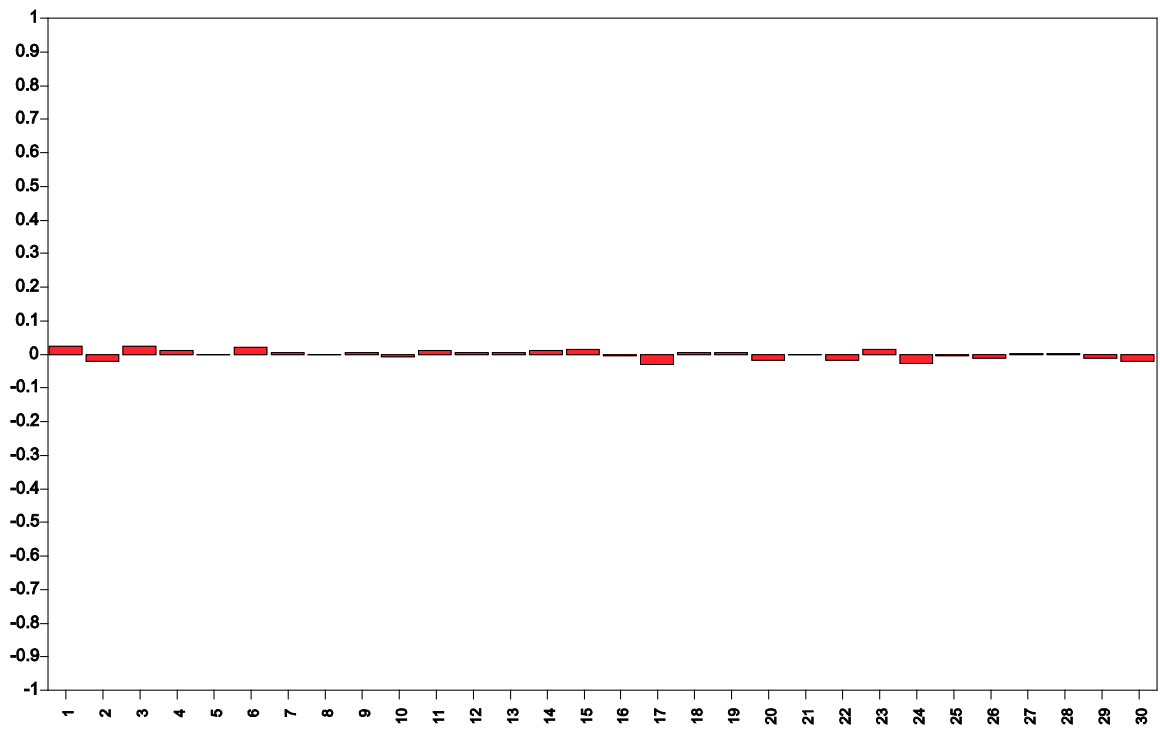


Figure 7: Indirect effect posterior from 10,000 iterations for the Firefighter example using priors

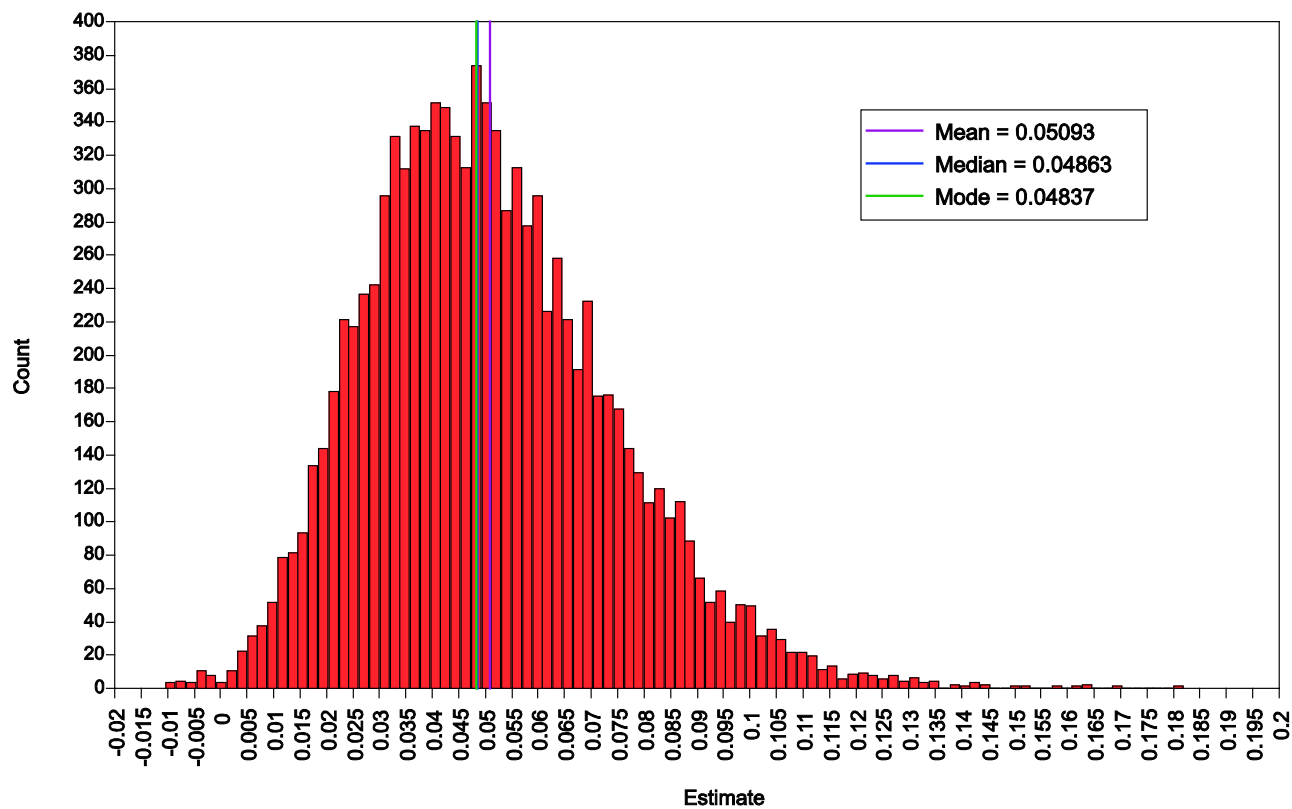


Figure 8: Posterior Predictive Checking (PPC) for Holzinger-Swineford data with simple structure CFA: PPC distribution

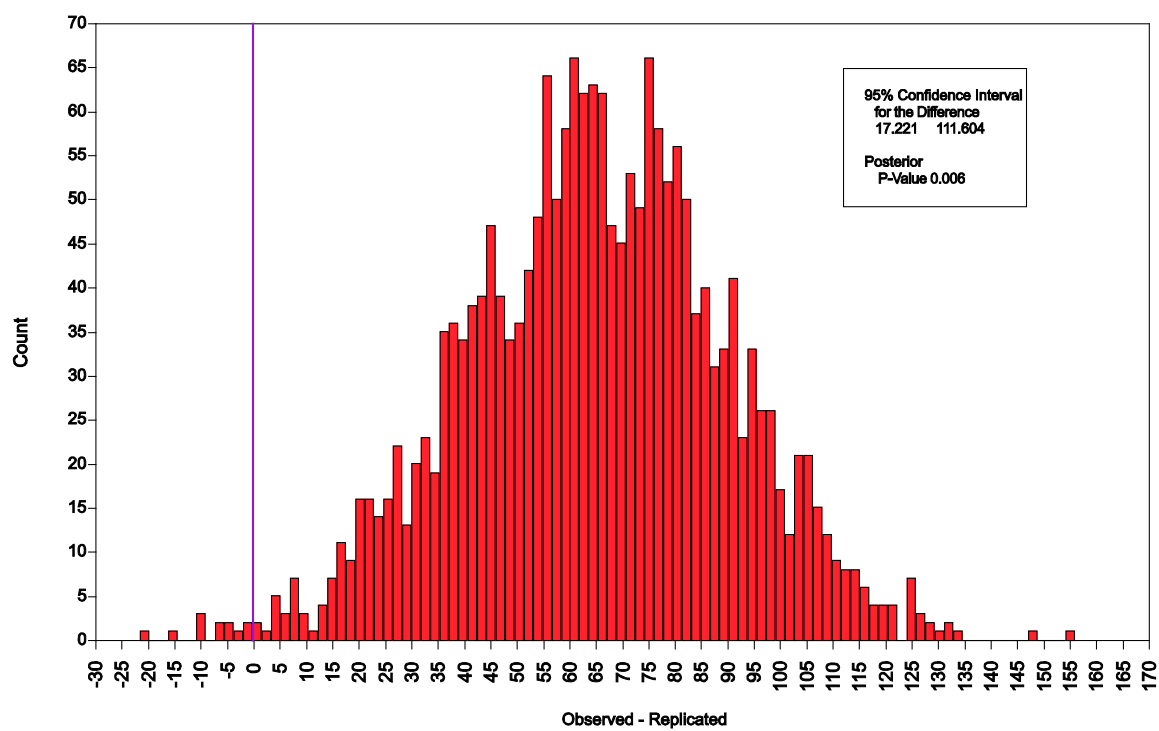
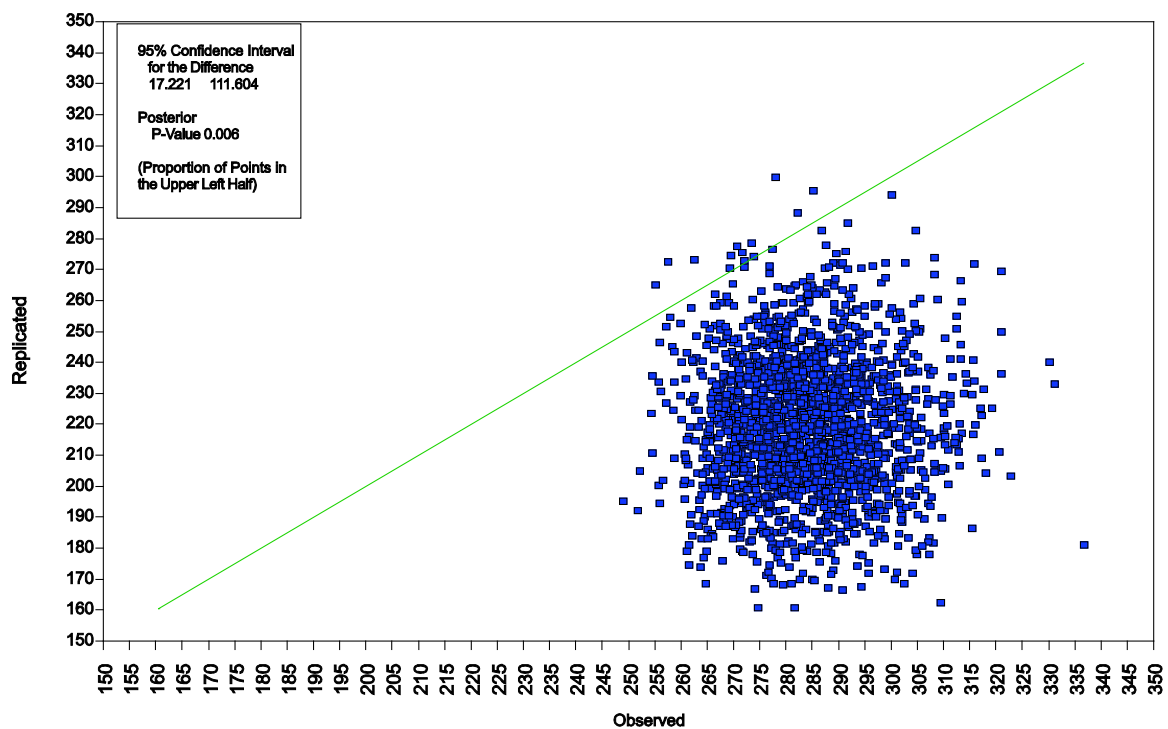
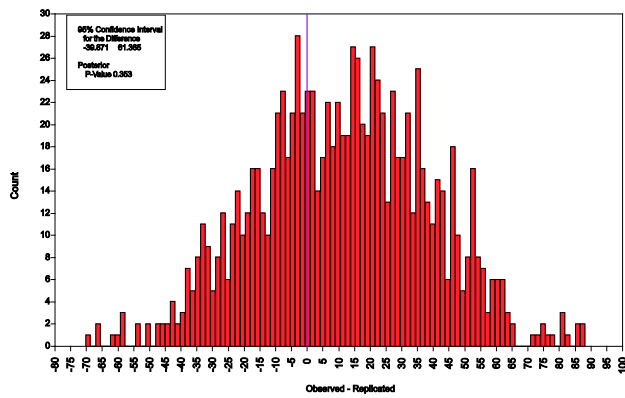
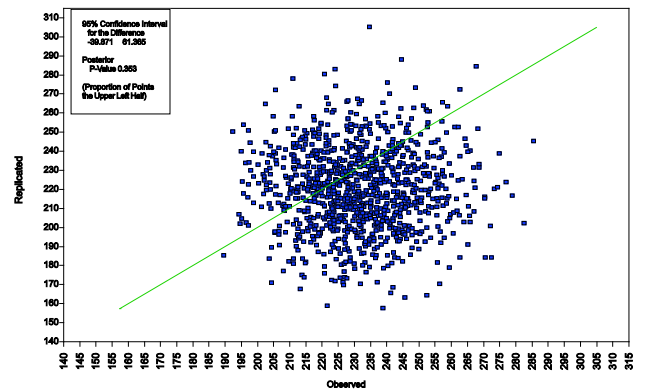


Figure 9: Posterior Predictive Checking (PPC) for Holzinger-Swineford data with simple structure CFA: PPC scatter plot





(a) PPC distribution for Bayes CFA



(b) PPC scatterplot for Bayes CFA

Figure 10: Posterior Predictive Checking (PPC) for Holzinger-Swineford data with priors on cross-loadings

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Table 1: Input for Bayes analysis of the Atlas example

TITLE:	ATLAS, Step 1
DATA:	FILE = mbr2004atlas.txt;
VARIABLE:	NAMES = obs group severity nutrit; USEV = group - nutrit;
ANALYSIS:	ESTIMATOR = BAYES; PROCESS = 2;
MODEL:	severity ON group (a); nutrit ON severity (b) group;
MODEL CONSTRAINT:	NEW (indirect); indirect = a*b;
OUTPUT:	TECH1 TECH8 STANDARDIZED;
PLOT:	TYPE = PLOT2;

Table 2: Output excerpts for Bayes analysis of the Atlas example

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I. Lower 2.5%	Upper 2.5%
severity ON					
group	0.282	0.106	0.010	0.095	0.486
nutrit ON					
severity	0.067	0.031	0.000	0.015	0.125
group	-0.011	0.089	0.440	-0.180	0.155
Intercepts					
severity	5.641	0.072	0.000	5.513	5.779
nutrit	3.698	0.191	0.000	3.309	4.108
Residual Variances					
severity	1.722	0.072	0.000	1.614	1.868
nutrit	1.331	0.070	0.000	1.198	1.468
New/Additional Parameters					
indirect	0.016	0.013	0.010	0.002	0.052

Table 3: TECH8 iterations with PSR for the first 1000 iterations of the 10,000 iterations analysis of the ATLAS example

Iteration	Potential Scale Reduction	Parameter With Highest PSR
100	1.037	2
200	1.014	4
300	1.002	2
400	1.003	3
500	1.002	7
600	1.002	6
700	1.000	6
800	1.003	1
900	1.002	1
1000	1.002	1

Table 4: Output excerpts from Bayesian analysis using 10,000 iterations for the ATLAS example

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
severity ON					
group	0.272	0.089	0.001	0.098	0.448
nutrit ON					
severity	0.074	0.030	0.008	0.014	0.133
group	-0.018	0.080	0.408	-0.177	0.140
Intercepts					
severity	5.648	0.062	0.000	5.525	5.768
nutrit	3.663	0.177	0.000	3.313	4.014
Residual Variances					
severity	1.719	0.083	0.000	1.566	1.895
nutrit	1.333	0.065	0.000	1.215	1.467
New/Additional Parameters					
indirect	0.019	0.011	0.009	0.003	0.045

Table 5: Input for Bayesian analysis of Firefighter example

TITLE:	Yuan and MacKinnon firefighters mediation using Bayesian analysis Elliot DL, Goldberg L, Kuehl KS, et al. The PHLAME Study: process and outcomes of 2 models of behavior change. J Occup Environ Med. 2007; 49(2): 204-213.
DATA:	FILE = fire.dat;
VARIABLE:	NAMES = y m x;
MODEL:	m ON x (a); y ON m (b) x;
ANALYSIS:	ESTIMATOR = BAYES; PROCESS = 2 ; FBITER = 10000;
MODEL CONSTRAINT:	NEW(indirect); indirect = a*b;
OUTPUT:	TECH1 TECH8;
PLOT:	TYPE = PLOT2;

Table 6: Output excerpts for Bayesian analysis of Firefighter example

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
m ON					
x	0.395	0.121	0.000	0.160	0.634
y ON					
m	0.142	0.052	0.003	0.040	0.243
x	0.108	0.117	0.176	-0.127	0.339
Intercepts					
y	0.418	0.057	0.000	0.308	0.530
m	0.000	0.059	0.499	-0.115	0.116
Residual Variances					
y	1.144	0.089	0.000	0.987	1.338
m	1.218	0.093	0.000	1.054	1.419
New/Additional Parameters					
indirect	0.053	0.028	0.004	0.011	0.117

Table 7: Input for Bayesian analysis with priors for Firefighter example

TITLE:	Yuan and MacKinnon firefighters mediation using Bayesian analysis Elliot DL, Goldberg L, Kuehl KS, et al. The PHLAME Study: process and outcomes of 2 models of behavior change. J Occup Environ Med. 2007; 49(2): 204-213.
DATA:	FILE = fire.dat;
VARIABLE:	NAMES = y m x;
MODEL:	m ON x (a); y ON m (b) x;
ANALYSIS:	ESTIMATOR = BAYES; PROCESS = 2; FBITER = 10000;
MODEL PRIORS:	a ~ N (0.35, 0.04); b ~ N (0.1, 0.01);
MODEL CONSTRAINT:	NEW(indirect); indirect = a*b;
OUTPUT:	TECH1 TECH8;
PLOT:	TYPE = PLOT2;

Table 8: Output excerpts for Bayesian analysis with priors for Firefighter example

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
m ON					
x	0.383	0.104	0.000	0.182	0.588
y ON					
m	0.133	0.046	0.003	0.042	0.223
x	0.112	0.117	0.169	-0.124	0.341
Intercepts					
y	0.418	0.056	0.000	0.308	0.530
m	0.000	0.059	0.499	-0.115	0.116
Residual Variances					
y	1.143	0.089	0.000	0.986	1.338
m	1.218	0.093	0.000	1.053	1.418
New/Additional Parameters					
indirect	0.049	0.023	0.003	0.012	0.102

Table 9: Holzinger-Swineford's hypothesized four domains measured by 19 tests

Factor loading pattern				
	Spatial	Verbal	Speed	Memory
visual	X	0	0	0
cubes	X	0	0	0
paper	X	0	0	0
flags	X	0	0	0
general	0	X	0	0
paragrap	0	X	0	0
sentence	0	X	0	0
wordc	0	X	0	0
wordm	0	X	0	0
addition	0	0	X	0
code	0	0	X	0
counting	0	0	X	0
straight	0	0	X	0
wordr	0	0	0	X
numberr	0	0	0	X
figurer	0	0	0	X
object	0	0	0	X
numberf	0	0	0	X
figurew	0	0	0	X

Table 10: Input for Bayes CFA

```

TITLE:      Bayes CFA on 19 variables from Holzinger-Swineford
DATA:      FILE = holzingr.dat;
           FORMAT = f3,2f2,f3,2f2/3x,13(1x,f3)/3x,13(1x,f3);
VARIABLE   NAMES = id female grade agey agem school visual
           cubes paper flags general paragrap sentence wordc wordm
           addition code counting straight wordr numberr figurer
           object numberf figurew deduct numeric problemr series
           arithmet paperrev flagssub; ! flags = lozenges
           USEV = visual cubes paper flags general paragrap sentence
           wordc wordm addition code counting straight wordr
           numberr figurer object numberf figurew;
           USEOBS = school EQ 0;
DEFINE:    visual = visual/sqrt(47.471);
           cubes = cubes/sqrt (19.622);
           paper = paper/sqrt(7.908);
           flags = flags/sqrt(68.695);
           general = general/sqrt(134.970);
           paragrap = paragrap/sqrt(11.315);
           sentence = sentence/sqrt(21.467);
           wordc = wordc/sqrt(28.505);
           wordm = wordm/sqrt(62.727);
           addition = addition/sqrt(561.692);
           code = code/sqrt(275.759);
           counting = counting/sqrt(437.752);
           straight = straight/sqrt(1362.158);
           wordr = wordr/sqrt(116.448);
           numberr = numberr/sqrt(56.496);
           figurer = figurer/sqrt(45.937);
           object = object/sqrt(20.730);
           numberf = numberf/sqrt(20.150);
           figurew = figurew/sqrt(12.845);
ANALYSIS:  ESTIMATOR = BAYES;
           PROCESS = 2;
           FBITER = 20000;
           STVAL = ML;
MODEL:     spatial BY visual* cubes paper flags;
           verbal BY general* paragrap sentence wordc wordm;
           speed BY addition* code counting straight;
           memory BY wordr* numberr figurer object numberf
           figurew;
           spatial-memory@1;
OUTPUT:    TECH1 TECH8 STDYX;
PLOT:     TYPE = PLOT2;

```

Table 11: Bayes CFA output excerpts: Model testing

Tests of model fit		
Bayesian Posterior Predictive Checking using Chi-Square		
95% Confidence Interval for the Difference Between the Observed and the Replicated Chi-Square Values		
	17.221	111.604
Posterior Predictive P-Value		0.006
Information Criterion		
Numbers of Free Parameters		63
Deviance (DIC)		6997.386
Estimated Number of Parameters (pD)		63.641

Table 12: Bayes CFA output excerpts: Model results

Parameter	Estimate	Posterior	One-Tailed	95% C.I.	
		S.D.	P-Value	Lower 2.5%	Upper 2.5%
spatial BY					
visual	0.721	0.093	0.000	0.543	0.905
cubes	0.497	0.098	0.000	0.306	0.690
paper	0.568	0.097	0.000	0.382	0.764
flags	0.700	0.096	0.000	0.514	0.892
verbal BY					
general	0.848	0.078	0.000	0.705	1.011
paragrap	0.861	0.078	0.000	0.717	1.021
sentence	0.878	0.077	0.000	0.738	1.039
wordc	0.730	0.082	0.000	0.578	0.900
wordm	0.884	0.077	0.000	0.743	1.043
speed BY					
addition	0.682	0.093	0.000	0.499	0.869
code	0.732	0.088	0.000	0.565	0.912
counting	0.732	0.089	0.000	0.562	0.910
straight	0.785	0.089	0.000	0.617	0.963
memory BY					
wordr	0.538	0.098	0.000	0.349	0.735
numberr	0.541	0.095	0.000	0.362	0.736
figurer	0.629	0.095	0.000	0.447	0.821
object	0.653	0.095	0.000	0.471	0.844
numberf	0.688	0.094	0.000	0.508	0.877
figurew	0.508	0.096	0.000	0.323	0.702
verbal WITH					
spatial	0.635	0.080	0.000	0.461	0.764
speed WITH					
spatial	0.632	0.103	0.000	0.392	0.787
verbal	0.496	0.087	0.000	0.309	0.651
memory WITH					
spatial	0.693	0.095	0.000	0.492	0.868
verbal	0.532	0.086	0.000	0.342	0.675
speed	0.653	0.081	0.000	0.482	0.792

Table 13: Input excerpts for Bayes CFA with informative priors on cross-loadings

```

ANALYSIS: ESTIMATOR = BAYES;
           PROCESS = 2;
           FBITER = 10000;
MODEL:    spatial BY visual* cubes paper flags;
           verbal BY general* paragrap sentence wordc wordm;
           speed BY addition* code counting straight;
           memory BY wordr* numberr figurer object numberf
           figurew;
           spatial-memory@1;
           spatial BY general-figurew*0 (a1-a15);
           verbal BY visual-flags*0 (b1-b4);
           verbal BY addition-figurew*0 (b5-b14);
           speed BY visual-wordm*0 (c1-c9);
           speed BY wordr-figurew*0 (c10-c15);
           memory BY visual-straight*0 (d1-d13);
MODEL PRIORS:
           a1-a15 ~ N(0,.01);
           b1-b14 ~ N(0,.01);
           c1-c15 ~ N(0,.01);
           d1-d13 ~ N(0,.01);
OUTPUT:   TECH1 TECH8 STDYX;
PLOT:     TYPE = PLOT2;

```

Table 14: Output excerpts for Bayes CFA with informative priors on cross-loadings

Tests of model fit		
Bayesian Posterior Predictive Checking using Chi-Square		
95% Confidence Interval for the Difference Between the Observed and the Replicated Chi-Square Values		
	-39.871	61.365
Posterior Predictive P-Value		0.353
Information Criterion		
Number of Free Parameters		120
Deviance (DIC)		6966.543
Estimated Number of Parameters (pD)		85.406

Table 15: Output excerpts for Bayes CFA with informative priors on cross-loadings

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
spatial BY					
visual	0.668	0.127	0.000	0.436	0.935
cubes	0.542	0.127	0.000	0.300	0.800
paper	0.480	0.125	0.000	0.245	0.731
flags	0.704	0.135	0.000	0.463	0.987
general	0.035	0.076	0.000	0.463	0.987
paragrap	-0.007	0.075	0.464	-0.116	0.183
sentence	-0.052	0.076	0.248	-0.203	0.095
wordc	0.051	0.075	0.252	-0.098	0.199
wordm	-0.018	0.075	0.404	-0.164	0.127
addition	-0.178	0.082	0.014	-0.337	-0.021
code	-0.005	0.078	0.474	-0.158	0.150
counting	0.010	0.082	0.447	-0.154	0.168
straight	0.190	0.079	0.010	0.029	0.339
wordr	-0.045	0.084	0.291	-0.215	0.112
numberr	-0.004	0.083	0.482	-0.169	0.156
figurer	0.123	0.083	0.070	-0.041	0.283
object	-0.151	0.082	0.031	-0.311	0.008
numberf	0.092	0.083	0.133	-0.076	0.252
figurew	0.009	0.083	0.456	-0.157	0.171

Table 16: Output excerpts for Bayes CFA with informative priors on cross-loadings (continued)

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
verbal WITH					
spatial	0.574	0.118	0.001	0.310	0.748
speed WITH					
spatial	0.511	0.132	0.000	0.199	0.704
verbal	0.492	0.107	0.000	0.246	0.662
memory WITH					
spatial	0.577	0.131	0.000	0.290	0.777
verbal	0.562	0.103	0.000	0.314	0.719
speed	0.609	0.098	0.000	0.378	0.766

Table 17: Input excerpts for Bayes CFA with three free cross-loadings

```
ANALYSIS: ESTIMATOR = BAYES;
           PROCESS = 2;
           FBITER = 20000;
MODEL:    spatial BY visual* cubes paper flags;
           verbal BY general* paragraf sentence wordc wordm;
           speed BY addition* code counting straight;
           memory BY wordr* numberr figurer object numberf
           figurew;
           spatial-memory@1;
           spatial BY addition straight object;
OUTPUT:   TECH1 TECH8 STDYX;
PLOT:     TYPE = PLOT2;
```

Table 18: Output excerpts for Bayes CFA with three free cross-loadings

Tests of model fit		
Bayesian Posterior Predictive Checking using Chi-Square		
95% Confidence Interval for the Difference Between the Observed and the Replicated Chi-Square Values		
	-26.253	70.376
Posterior Predictive P-Value		0.173
Information Criterion		
Number of Free Parameters		66
Deviance (DIC)		6957.757
Estimated Number of Parameters (pD)		65.223

Table 19: Output excerpts for Bayes CFA with three free cross-loadings (continued)

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
spatial BY					
visual	0.768	0.090	0.000	0.598	0.949
cubes	0.511	0.097	0.000	0.325	0.709
paper	0.583	0.095	0.000	0.398	0.771
flags	0.701	0.091	0.000	0.526	0.886
addition	-0.402	0.171	0.001	-0.834	-0.136
straight	0.349	0.104	0.001	0.138	0.550
object	-0.660	0.272	0.000	-1.317	-0.273
verbal BY					
general	0.851	0.078	0.000	0.707	1.013
paragrap	0.864	0.077	0.000	0.722	1.022
sentence	0.883	0.077	0.000	0.742	1.042
wordc	0.734	0.082	0.000	0.580	0.903
wordm	0.888	0.077	0.000	0.745	1.047
speed BY					
addition	1.024	0.167	0.000	0.757	1.430
code	0.705	0.089	0.000	0.537	0.884
counting	0.748	0.086	0.000	0.588	0.925
straight	0.544	0.102	0.000	0.349	0.750
memory BY					
wordr	0.513	0.095	0.000	0.331	0.703
numberr	0.514	0.095	0.000	0.332	0.705
figurer	0.620	0.093	0.000	0.445	0.807
object	1.208	0.268	0.000	0.829	1.849
numberf	0.682	0.089	0.000	0.515	0.865
figurew	0.499	0.095	0.000	0.316	0.692

Table 20: Output excerpts for Bayes CFA with three free cross-loadings (continued)

Parameter	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.	
				Lower 2.5%	Upper 2.5%
verbal WITH					
spatial	0.575	0.076	0.000	0.407	0.708
speed WITH					
spatial	0.598	0.101	0.000	0.379	0.769
verbal	0.505	0.087	0.000	0.314	0.657
memory WITH					
spatial	0.752	0.082	0.000	0.559	0.878
verbal	0.586	0.077	0.000	0.414	0.713
speed	0.682	0.074	0.000	0.507	0.800

Table 21: Model testing results for Holzinger-Swineford data ($n = 145$)

Model	DIC	PPC 95% CI	PPC p-value	ML p-value
1. CFA	6997	17 - 112	0.006	0.0002
2. CFA + priors	6967	-40 - 61	0.353	-
3. CFA + 3 free	6958	-26 - 70	0.173	0.049
4. CFA + 3 free + priors	6955	-49 - 51	0.478	-