

**Target Rotations and Assessing the Impact of Model Violations
on the Parameters of Unidimensional Item Response Theory Models**

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Target Rotations

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Abstract

Reise, Cook, and Moore (under review) proposed a “comparison modeling” approach to assess the distortion in item parameter estimates when a unidimensional item response theory (IRT) model is imposed on multidimensional data. Central to their approach is the comparison of item slope parameter estimates from a unidimensional IRT model (a restricted model), with the item slope parameter estimates from the general factor in an exploratory bifactor IRT model (the unrestricted comparison model). In turn, these authors suggested that the unrestricted comparison bifactor model be derived from a target factor rotation (Browne, 2001). The goal of this study was to provide further empirical support for the use of target rotations as a method for deriving a comparison model. Specifically, we conducted Monte Carlo analyses exploring: a) the use of the Schmid-Leiman (Schmid & Leiman, 1957) orthogonalization to specify a viable initial target matrix, and b) the recovery of true bifactor pattern matrices using target rotations as implemented in MPLUS (Asparouhov & Muthén, 2008). Results suggest that to the degree that item response data conform to independent cluster structure, target rotations can be used productively to establish a plausible comparison model.

Target Rotations and Assessing the Impact of Model Violations
on the Parameters of Unidimensional Item Response Theory Models

Most applications of item response theory (IRT) modeling have focused on unidimensional models (i.e., models that assume a single latent trait). Yet, if more than one latent trait is required to achieve local independence, then using a unidimensional model will result in biased IRT item parameters. There is a large literature that has explored the robustness of unidimensional IRT model parameter estimates to unidimensionality violations (e.g., Drasgow & Parsons, 1983; Reckase, 1979). One conclusion arising from this literature is that if factor analysis or a related method is applied to the data and a “strong” common factor, or multiple highly correlated factors are obtained, then unidimensional IRT item parameter estimates are reasonably robust, that is, they are not seriously distorted.

Drawing from the IRT robustness literature, many researchers have adopted the idea that it is safe to apply unidimensional IRT models when data have a “strong” common dimension. Consequently, researchers considering the application of unidimensional IRT models rely almost exclusively on rule-of-thumb guidelines (e.g., ratio of 1st to 2nd eigenvalues greater than 3) and post-hoc model analysis (e.g., small residuals after fitting a single factor, or acceptable “fit” to a one-factor model; McDonald & Mok, 1995) to judge whether item response data are “unidimensional enough” for application of IRT.

Reise, Cook, and Moore (under review), on the other hand, argued that no matter how reasonable, intuitive, or statistically rigorous, no rule-of-thumb index can provide an estimate of the amount of bias that exists in the item parameter estimates obtained by fitting a unidimensional IRT model when multidimensionality is present. Thus, as a complement to rule-

of-thumb guidelines and post-hoc model checking, they suggested fitting a multidimensional and a unidimensional IRT model and then comparing the resulting item slope parameters (see also Ip, 2010). For this comparison to be meaningful, the multidimensional IRT model must consist of an overall general dimension (to capture what is in common among the items) plus some additional group dimensions (to capture local dependencies). Furthermore, the comparison must be performed using an intercept/slope parameterization, not a threshold/factor loading parameterization, for reasons that will be discussed shortly.

Now, in order to obtain a multidimensional IRT solution with an overall dimension and specific additional dimensions (i.e., a hierarchical solution) requires some work. One first needs to find a plausible multidimensional model for the data at hand. Then, the resulting solution is transformed into a hierarchical solution using a Schmid-Leiman transformation (SL; Schmid & Leiman, 1957). However, as we shall show, the resulting SL solution may yield biased estimates of the parameters on the general dimension and should not be compared with the parameters estimated in the unidimensional model. This is because the SL solution leads to a constrained hierarchical solution. There are restrictions among the estimated parameters (see Yuan, Thissen & McLeod, 1999). Thus, an additional step is needed: using the SL solution (i.e., the constrained hierarchical solution) as target, a target rotation needs to be performed on the parameters of the multidimensional solution. The resulting solution is an unconstrained hierarchical model whose parameters on the general dimension are to be compared with those obtained in the unidimensional solution. Notice that if in the unconstrained solution each item depends on just the overall factor and a group factor (regardless of the number of group factors) this is a bifactor model (Gibbons & Hedeker, 1992) and throughout this paper we shall use the term bifactor and unconstrained hierarchical model interchangeably.

Obviously, finding the best multidimensional IRT model may be rather computer intensive and one may question the utility of so much computational effort simply to investigate the robustness of IRT parameter estimates. However, the multidimensional IRT model can be estimated using limited information methods (from tetrachoric/polychoric correlations) and as a result, the full procedure may take mere seconds. Note that when estimating the model in this fashion, thresholds and factor loadings are estimated. Ultimately, these need to be transformed to intercepts and slopes for comparison to the unidimensional results, as described below.

Reise, Cook, and Moore (under review) have summarize the steps needed to investigate the robustness of unidimensional IRT parameter estimates to local independence violations as follows:

- 1) Find the best fitting and plausible multidimensional factor model for the data at hand estimating the model using tetrachoric/polychoric correlations.
- 2) Apply a Schmid-Leiman (SL) transformation to the factor loadings obtained in 1) to obtain a restricted hierarchical solution.
- 3) Using the solution obtained in 2) as target, obtain an unrestricted hierarchical solution using target rotations (Browne, 2001).
- 4) Transform the factor loadings obtained in 3) to IRT slopes. Compare them with the slopes obtained for a unidimensional solution. It is preferable that for this comparison the parameters of the unidimensional solution are also obtained using limited information methods (i.e., estimating the model using tetrachoric/polychoric correlations) to avoid possible biases in the comparison induced by the use of different estimation methods.

Reise, Cook, and Moore (under review) suggested that a finding of little or no difference in IRT slopes argues for the application of a unidimensional IRT model while a finding of large

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3 differences argues for: a) fitting a multidimensional IRT model -- a bifactor IRT model (Gibbons
4 & Hedeker, 1992), a unidimensional model with local dependencies modeled through correlated
5 residuals (Ip, 2010), or an independent clusters solution (McDonald, 1999; 2000), b) revising the
6 scale by deleting problematic items from the analyses (e.g., items that load on more than one
7 group factor in a hierarchical solution model), or c) abandoning any latent variable model (i.e.,
8 concluding the scale lacks any identifiable and interpretable latent structure).

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18 In this study, our primary goal is to elaborate on and provide empirical support for the
19 approach suggested in Reise, Cook, and Moore (under review). To do so, first we provide an
20 example to illustrate their proposal. Two sections follow the numerical example: First, we review
21 the relations between ordinal factor analysis (OFA) and IRT models in order to justify our use of
22 OFA for studying the parameters of IRT models. Second, Monte Carlo simulation is used to
23 explore: a) the viability of using a SL transformation to suggest an initial bifactor target matrix
24 (steps 2 and 3), and b) assuming that the target pattern matrix is specified correctly, the ability of
25 targeted bifactor pattern rotations to recover true population parameters.

36 37 38 39 **Ordinal Factor Analysis and Item Response Theory**

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41 In this section, we review the different parameterizations that may be used in two-
42 parameter IRT models. These are well known and have received considerable treatment in the
43 psychometric literature (e.g., Takane, & de Leeuw, 1987; Knol & Berger, 1991; McDonald &
44 Mok, 1995; Bartholomew & Knott, 1999; McDonald, 2000; McLeod, Swygert, & Thissen, 2001;
45 Ackerman, 2005; Wirth & Edwards, 2007; Forero & Maydeu-Olivares, 2009). However, as we
46 shall see, in the case of bifactor models equivalent bifactor models can appear very different in
47 terms of the implied data structure just because of the choice of parameterization.

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Item Response Theory Models

IRT models specify an item response curve (IRC) to characterize the non-linear relation between individual differences on a continuous latent variable (θ) and the probability of endorsing a dichotomous item x_i . Equation 1 shows the two-parameter normal-ogive model:

$$P(x_i = 1 | \theta) = E(x_i | \theta) = \int_{-\infty}^{z_i} \frac{1}{\sqrt{2\pi}} \exp(-Z_i^2 / 2) dt . \quad (1)$$

In Equation 1, z is a normal deviate equal to $\alpha(\theta - \beta)$, where α is an item discrimination parameter and β is an item location parameter corresponding to where on the latent trait scale the probability of endorsing the item is .50. Stated in different terms, Equation 1 is a cumulative normal distribution with a mean equal to β and standard deviation $\sigma = 1 / \alpha$. Thus, $z = (\theta - \mu) / \sigma$ is simply the z -score derived from a particular normal distribution which is used to determine the cumulative proportion of cases that fall at or below the normal deviate.

A more commonly applied IRT model that yields a very similar fit is the two-parameter logistic model (2PL) shown in Equation 2.

$$P(x_i = 1 | \theta) = E(x_i | \theta) = \frac{\exp(1.7z_i)}{1 + \exp(1.7z_i)} = \frac{\exp(1.7\alpha_i(\theta - \beta_i))}{1 + \exp(1.7\alpha_i(\theta - \beta_i))} . \quad (2)$$

where 1.7 is a scaling factor that makes the item slope parameter comparable to its value under the normal-ogive parameterization (Equation 1). Including the 1.7 is critical because only the normal-ogive parameterization can be easily linked with ordinal factor analytic terms.

Equations 1 and 2 express the two-parameter model using its most commonly found parameterization. However, this is not the parameterization that is used by most IRT software packages, and this parameterization is not readily generalizable to the multidimensional case. Instead it is better to write,

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$$z_i = \alpha_i \theta - \alpha_i \beta_i, \quad (3)$$

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6 and then define an intercept term $\gamma = -\alpha\beta$ and rewrite the deviate as shown in Equation 4.

$$z_i = \alpha_i \theta + \gamma_i. \quad (4)$$

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11 Obviously, because the slope is always positive, items with negative locations (easy items) will
12 have positive intercepts and items with positive locations (difficult items) will have negative
13 intercepts. Thus γ reflects the relative "easiness" of the item.

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19 Equation 4 has a straightforward generalization to the multivariate case, where the
20 number of dimensions is $p = 1 \dots P$ (possibly) correlated factors.

$$z_i = \sum \alpha_{ip} \theta_p + \gamma_i. \quad (5)$$

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26 In Equation 5, γ is now a multidimensional intercept parameter representing the deviate
27 corresponding to the expected proportion endorsing an item for individuals who are at the mean
28 on all latent traits. In the simple case of a bifactor IRT model (e.g., Gibbons & Hedeker, 1992)
29 where each item loads on a single general factor (G), and at most on one additional orthogonal
30 group factor ($p' =$ relevant group factor), Equation 5 becomes:

$$z_i = \alpha_{iG}(\theta_G) + \alpha_{ip'}(\theta_{p'}) + \gamma_i. \quad (6)$$

43 Ordinal Factor Analytic Models

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46 Applying the usual (linear) factor analysis model to dichotomous or polytomous items is
47 problematic for a variety of reasons (see, Bernstein & Teng, 1989; Wirth & Edwards, 2007).

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49 However, the model can be easily modified to accommodate ordinal responses by adding a
50 threshold process to the linear factor model for continuously distributed responses

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53 (Christofferson, 1975; Muthén, 1979; Maydeu-Olivares, 2006). For example, consider the
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55 binary case and n items. The ordinal factor analysis model assumes that the observed 0 or 1 item
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response is a discrete realization of a continuous and normally distributed latent response process (x^*) underlying the items. More specifically, consider the fully standardized model:

$$E(x_i^* | \theta_1 \dots \theta_p) = x_i^* = \lambda_{i1}\theta_1 + \lambda_{i2}\theta_2 + \lambda_{i3}\theta_3 + \dots + \lambda_{ip}\theta_p + \varepsilon_i . \quad (7)$$

In this model, x^* is not an observed 0 or 1 item score but rather is a continuous and normally distributed response propensity variable assumed to underlie item performance. In addition, λ is a factor loading (correlation) which, in the unidimensional case, can be shown to be analogous to an item-test biserial correlation. To develop the model further, an item threshold parameter (τ) needs to be specified such that $x = 1$, if $x^* \geq t_i$ and $x_i = 0$, if $x^* < t_i$. That is, an individual will endorse an item only if his/her response propensity is above the item's threshold.

The model in Equation 7 assumes that the errors are multivariate normal and uncorrelated with each other and the latent traits. The implied correlation matrix is thus:

$$\Sigma = \Lambda\Phi\Lambda' + \Psi . \quad (8)$$

where, Σ is the implied correlation matrix for the "hypothetical" latent propensities (i.e., it is a tetrachoric correlation matrix), Λ is an $n \times p$ matrix of factor loadings, Φ is a $p \times p$ matrix of factor correlations, and Ψ is a $p \times p$ diagonal matrix of residual variances such that $\Psi = \mathbf{I} - \text{diag}(\Lambda\Phi\Lambda')$.

It turns out that the OFA model just described and the normal ogive model in Equation (1) are equivalent models. Indeed, we can write the normal deviate for the multidimensional or bifactor cases:

$$z_i = \frac{\lambda_{i1}\theta_1 + \lambda_{i2}\theta_2 + \dots + \lambda_{ip}\theta_p - \tau_i}{\sqrt{1 - \sum \lambda_i^2}} \quad \text{or,} \quad z_i = \frac{\lambda_{iG}\theta_G + \lambda_{ip}\theta_p - \tau_i}{\sqrt{1 - \sum \lambda_i^2}} . \quad (9)$$

Clearly, in either case the normal deviate is formed as a weighted linear combination of the factor loadings, which is then scaled relative to the square root of an item's unique variance (the

residual standard deviation of x^*). Finally, when either of the normal deviates in Equation 9 is substituted into Equation 1, the model becomes a normal-ogive OFA model.

Transformations

The equations to translate between an OFA and the two-parameter normal-ogive IRT model are cited in dozens of readily available sources (e.g., Knol & Berger, 1991; McDonald, 2000, McLeod, Swygert & Thissen, 2001; Ackerman, 2005), and are obvious through inspection of Equation 9. Assuming that the latent factors are uncorrelated, the translations between FA loadings and IRT slopes (normal ogive metric) are:

$$\lambda_{ip} = \frac{\alpha_{ip}}{\sqrt{1 + \sum_{p=1}^P \alpha_{ip}^2}} \quad \alpha_{ip} = \frac{\lambda_{ip}}{\sqrt{\psi_i}} = \frac{\lambda_{ip}}{\sqrt{1 - \sum_{p=1}^P \lambda_{ip}^2}} . \quad (10)$$

The denominator in the latter equation is readily interpretable as the square root of the item's uniqueness, ψ_i , (one minus the communality). Thus, it is immediately apparent that a slope α on a particular dimension is a function of both an item's loading on that dimension, λ , and its uniqueness, ψ , within the context of a specific model (Muthén & Christofferson, 1981, p. 411); items with more common variance in the OFA model will have relatively higher slopes in the IRT parameterization. This is exactly why the interpretation of equivalent multidimensional IRT and OFA models is challenging, as we will show shortly.

Note that the denominator of either of the above equations becomes considerably more complicated in the case of multiple correlated dimensions. For example, in the case of two correlated factors, the denominator for the IRT slope equation is:

$$\sqrt{\psi_i} = \sqrt{1 - \sum_{p=1}^P \lambda_{ip}^2 - 2\lambda_1\lambda_2\phi_{12}} . \quad (11)$$

Finally, the translation between IRT item intercepts and factor analytic thresholds follows a similar logic as Equation 10.

$$\tau_i = \frac{-\gamma_i}{\sqrt{1 + \sum_{p=1}^P \alpha_{ip}^2}} \quad \gamma_i = \frac{-\tau_i}{\sqrt{1 - \sum_{p=1}^P \lambda_{ip}^2}} . \quad (12)$$

Interpreting Multidimensional Solutions in OFA and IRT

When data are strictly unidimensional the relations between OFA and IRT are simple because the item parameter estimates leave the researcher with much the same impression of the data structure and psychometric functioning of the items. For example, in the first column in the top half of Table 1 are factor loadings for 15 items. This matrix mimics a hypothetical test with three content clusters of five items each, where the relation between the item and common latent trait is equal within cluster but not between. In the next column are the transformed IRT parameters. Clearly, the same message about the data structure is conveyed – items 1 through 5 are the best, items 11 through 15 are the worst, and the relations between item and latent trait are the same within cluster but not between.

Such symmetry of translation is, generally speaking, not possible when data have a multidimensional structure and the items vary in communality. In the top middle portion of Table 1 we display the loading matrix for a bifactor model with one general and three orthogonal group factors. The items are all equally related to the general trait, but the content clusters differ in their loadings on the group factors. In the final set of columns are shown the equivalent transformed IRT model parameters. The items do not have equal relations to the general trait, but rather the subset of items with the smallest uniqueness appear to be the best items in terms of measuring the general factor. In short, the factor analytic parameterization suggests a different psychometric interpretation than under the IRT parameterization.

In the bottom portion of Table 1 we reversed the process by starting with an IRT model with equal slopes on the general factor and varying the group factor slopes. Specifically, the item slopes on the general factor are 0.75, and group slopes are 0.58, 0.32, and 0.10. The true slopes in the bottom portion of Table 1 mimic the true factor values in the top half of Table 1 in that these are the IRT slopes equal to the factor loadings ignoring the multidimensionality. That is, the true slopes are simply the true loadings divided by the square root of 1 minus the loading squared. Clearly, the right hand side in the lower half of Table 1 demonstrates that when IRT parameters are translated to OFA terms, the items no longer load on the general factor equally.

The results in Table 1 demonstrate that although the IRT and OFA models are equivalent in implied correlational structure, in the multidimensional case there is a non-symmetry to the interpretation due to the critical role of an item's uniqueness. This non-symmetry of interpretation must be considered when a researcher is using the results of one parameterization to understand what the equivalent model would look like under the alternative parameterization. Fortunately, programs for estimating multidimensional models such as TESTFACT (Bock et al., 2002) and NOHARM (Fraser, 1988; Fraser & McDonald, 1988) routinely provide results in both IRT and OFA terms. Nevertheless, this non-symmetry of interpretation in multidimensional models can cause confusion when interpreting the results of research into the effects of model violations on item parameter estimates.

Parameter estimation in OFA and IRT

The above outlined the theoretical relations between OFA and IRT parameters; however, in practice, such parameters need to be estimated. Generally speaking, there are two strategies for estimating these parameters. One strategy, used in OFA, estimates the parameters using only bivariate information, in several stages, generally through the use of tetrachoric (or polychoric)

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3 correlations (see for instance Muthén, 1993). The other strategy, used in IRT estimates the
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5 parameters using full information maximum likelihood estimation (Bock & Aitkin, 1981). Thus,
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7 even when the same model is estimated, as in the case of the normal ogive model, OFA and IRT
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9 use slightly different approaches. In OFA, the threshold and factor loading parameterization is
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11 used, and parameters are estimated using bivariate information. In IRT, the intercept and slope
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13 parameterization is used, and parameters are estimated using full information maximum
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15 likelihood. Full information maximum likelihood is theoretically superior to bivariate estimation
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17 methods, but the latter are considerably faster for multidimensional models. Another advantage
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19 of the bivariate information methods is that a goodness-of-fit test of the model to the tetrachoric
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21 correlations is available (see Muthén, 1993; Maydeu-Olivares, 2006) – and an associated
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23 RMSEA. In contrast, comparable procedures for goodness-of-fit assessment in the case of full
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25 information maximum likelihood estimation (e.g. Maydeu-Olivares & Joe, 2005, 2006) are not
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27 yet available in standard software.
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34 Reise, Cook, and Moore (under review) argued that the simplest bivariate estimation
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36 method, common factoring via unweighted least squares estimation of a tetrachoric matrix is a
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38 good enough approach for the purposes of determining the robustness of unidimensional
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40 modeling of multidimensional data. This view is supported by research such as Forero and
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42 Maydeu-Olivares (2009), who performed an exhaustive Monte Carlo comparison of full and
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44 bivariate estimation strategies and concluded that for samples over 500 observations the
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46 differences in parameter accuracy between both methods were negligible (and even slightly
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48 favoring unweighted least squares). In the next section we consider the development and
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50 specification of a comparison bifactor model using the SL and target rotations.
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The Schmid-Leiman and Target Bifactor Rotations

As described above, in the comparison modeling approach, estimating the degree to which model violations affect unidimensional IRT item parameter estimates involves the comparison of an exploratory bifactor model (i.e., the unrestricted model) with a highly restricted model (i.e., a unidimensional model). Although the model comparison idea is simple enough, the realization in practice is challenging due to the fact that commonly used analytic rotations are geared towards identifying simple structure solutions (see Browne, 2001 for detailed discussion). When the true data structure is bifactor in the population, exploratory analytic rotations such as promax or oblimin will fail to identify the correct underlying data structure.

It must be recognized that in practice, in order to use exploratory multidimensional models to judge the effects of forcing data into a restricted model, the comparison model must be plausible and reasonably reflect the population loading pattern. That is, it must approximate what the true factor loadings or IRT slopes would be without the imposed restrictions; one cannot meaningfully compare a wrong bifactor model with a more wrong unidimensional model. The challenge is, given the limitations of analytic rotations, how can researchers identify an acceptable exploratory method that can identify a bifactor structure?

As noted in the steps above, Reise, Cook, and Moore (under review) suggested that a SL (Schmid & Leiman, 1957) orthogonalization be used as a basis for specifying a bifactor target pattern, and then a target pattern rotation (Browne, 2001) be used to estimate an unrestricted comparison model. This begs two questions. First, assuming that in the population the data structure is bifactor, under what conditions can the SL orthogonalization identify a good target matrix? By good target matrix, we mean a matrix that specifies all near zero loadings to specified (0), and all non-trivial loadings to unspecified (?). Second, given a correct target

matrix, how well can target rotations recover true population parameters? In the following two sections, we use Monte Carlo simulation to address each of these issues in turn.

The Schmid-Leiman: Getting the Target Bifactor Pattern Right

Target pattern rotations require an initial target matrix of specified (0) and unspecified (?) elements. “This target matrix reflects partial knowledge as to what the factor pattern should be” (Browne, 2001, p. 124). In addition, the target matrix forms the basis for a rotation that minimizes the sum of squared differences between the target and the rotated factor pattern. Although target rotations potentially have a self-correcting element (i.e., just because a loading is specified as zero does not mean the loading estimate must be zero and vice versa), the robustness of misspecifying the elements of a target matrix is currently unknown. Thus, at this point, target rotations (Browne, 2001) can only be expected to function optimally when the target pattern is correctly specified.

It follows that an important goal of comparison modeling is to identify a bifactor structure so that a researcher can form a target matrix such that items with non-trivial (e.g., $\geq .30$) loadings on the group factors in the population are unspecified (?), and items with trivial or near zero loadings are specified (0). Note that it is assumed all items have non-trivial loadings on the general factor. Of course, there are many ways to specify an initial target matrix. For example, a target can be generated from theory, or from exploratory data analyses such as factor analysis, cluster analyses, or multidimensional scaling.

In conjunction with these methods, Reise, Cook, and Moore (under review) suggested that the SL orthogonalization be used as an exploratory tool to determine the number of identified group factors in a data matrix and to determine the pattern of trivial and non-trivial

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3 loadings on the group factors. Here, we define a non-trivial loading as being greater than or
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5 equal to .15 in a SL (Schmid & Leiman, 1957) solution (as opposed to a population value) for
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7 reasons that will be made clear shortly. In this section, we consider the ability of the SL to
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9 identify an appropriate target matrix.
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13 The SL is an orthogonalization of a higher-order factor model, which itself is a product of
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15 a common factor analysis with oblique rotation (McDonald, 2000). For example, starting with a
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17 tetrachoric correlation matrix, a researcher: a) extracts (e.g., unweighted least squares) a
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19 specified number of primary factors and rotates the factors obliquely (e.g., oblimin), and b)
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21 factor analyzes the primary factor correlation matrix in order to determine the relation between
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23 each primary factor and a general factor. Given this higher-order representation, the SL is easily
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25 obtained. Specifically, an item's loading on the general factor is simply its loading on the
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27 primary factor multiplied by the loading of the primary factor on the general factor. An item's
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29 loading on a group factor is its loading on the primary factor multiplied by the square root of the
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31 disturbance (i.e., that part of the primary factor that is not explained by the general factor).
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37 The Achilles heel of the SL procedure is that because of the well known proportionality
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39 constraints (Yung, Thissen, & McLeod, 1999), the SL estimates of population loadings are
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41 unbiased under only a narrow range of conditions. The proportionality constraints emerge
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43 because, as described above, for each item, the group and general loadings in the SL are
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45 functions of common elements (i.e., the loading of the primary on the general and the unique
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47 variance of the primary). As a consequence, a researcher cannot rely on the SL to derive an
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49 acceptable comparison model due to the bias in factor loadings that emerge under some
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51 conditions.
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3 To demonstrate this, in the first set of columns in the top portion of Table 2, we specified
4 a population bifactor structure with equal loadings on the general and equal loadings within each
5 group factor. In the bottom portion is a SL of the implied correlation matrix, extracting three
6 factors using unweighted least squares and rotating using oblimin. The SCHMID routine
7 included in the PSYCH package (Revelle, 2009) of the *R* software program (*R* Development
8 Core Team, 2008) was used for all analyses. Obviously, in this example, the SL perfectly reflects
9 the population structure because the loadings within the group factors are proportional to the
10 general factor. If real data were to always mimic this perfectly proportional structure, the SL
11 would always yield the correct comparison matrix (given enough data) and there would be no
12 need for target bifactor rotations.
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27 In the top portion of Table 2 in the middle set of columns we specified an extreme
28 condition where loadings on the general are all .40 but loadings within each group factor were
29 selected from a random uniform distribution ranging from 0.10 to 0.70. Note that this structure
30 is perfect independent cluster (IC; McDonald, 1999; 2000), that is, there are no items with cross-
31 loadings on the group factors (which is equivalent to saying there are no items with cross-
32 loadings in the oblimin solution), and each orthogonal group factor has three items to identify it.
33 This particular true loading pattern, although probably unrealistic in practice, was selected
34 because of its clear violations of proportionality.
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46 After converting these loadings into a correlation matrix, in the bottom portion of Table 2
47 in the middle set of columns are shown the SL loadings for this independent clusters data.
48 Consider first group factor one (items one through five) where for three of the five items, the true
49 group factor loadings are higher than the true general factor loadings, and for the remaining two
50 items, the group factor loadings are way below the loadings for the general. For the latter two
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3 items, in the SL the loadings on the general factor are underestimated, and the loadings on the
4
5 group factor are overestimated. In addition, small positive loadings occur for these items on
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7 group factors two and three. For the items in group factor two, items six and seven have
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9 overestimated general factor loadings and underestimated group factor loadings. The reverse
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11 occurs for items eight through ten.
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15 Such findings are easily anticipated from inspection of the original communalities and
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17 inspection of the loadings in an oblique rotated factor pattern. However, the important lesson is
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19 that although the SL can reasonably reproduce the correct pattern of trivial and non-trivial
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21 loadings even under these extreme conditions, the specific factor loadings in the SL are not
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23 proper estimates of their corresponding population values. This result can be entirely attributed
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25 to the proportionality constraints, and implies that the SL loadings should not, generally
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27 speaking, be used as a comparison matrix.
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32 Finally, in the third set of columns we have reproduced the population loading pattern
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34 from the first demonstration but added large (.50) cross-loadings for items one, six, and eleven.
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36 The data are no longer independent cluster but rather have an independent cluster basis (ICB;
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38 each group factor is identified by three items with simple-loadings, but there are also items with
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40 cross-loadings). In the bottom portion we see the effects of cross-loadings on the SL results.
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42 Specifically, a cross-loading raises an item's communality and thus causes an item's loading on
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44 the general factor to increase. This overestimation is proportional to the communality of the item.
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46 In turn, loadings on the group factors decrease relative to their true population values. The
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48 critical lesson learned from this demonstration is that in the SL, one must proceed very
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50 cautiously in interpreting the size of the loadings, and they should not be taken seriously as
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52 estimates of population parameters.
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Target Rotations

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3 The above demonstrations do not necessarily spoil our ability to use the SL to identify an
4 appropriate target matrix. In specifying a target pattern a researcher does not have to get the
5 loadings right but only the pattern of trivial and non-trivial loadings. However, the fact that
6 loadings in the SL are not good estimates of their population values must be considered in
7 judging how to specify an initial target in real data situations. A topic we now address.
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11 As noted, we are interested in identifying any group factor loading that is greater than or
12 equal to .30 in the population (we always assume items on the general meet this criterion). To
13 accomplish this, we use the SL results to build a target pattern by: a) specifying that all items are
14 unspecified (?) on the general factor (if preliminary analyses suggest otherwise, we delete the
15 item) and on the group factors, b) if the SL loading is greater than or equal to .15 the element is
16 unspecified, otherwise it is specified. The reason we selected this liberal criterion is that we are
17 most concerned with not missing any cross-loadings if they exist in the population; if an item has
18 a sizable loading on the general and group factor, as well as a cross-loading, its communality will
19 tend to be high and its group factor loadings underestimated in the SL. To allow for this
20 potential underestimation we selected the liberal .15 criterion.
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39 This decision rule begs the question, how well does it work in practice? Even without
40 data, it is obvious that the largest problems will occur with loadings that are around .30 in the
41 population. Depending on sample size, and the distortions of the SL, such values may be
42 underestimated in real data and thus incorrectly specified. The other, less serious problem is that
43 of loadings that are truly zero in the population, but overestimated in the SL using real data. This
44 would cause the error of not specifying an element when it truly should be specified. Because of
45 limited research on the effects of incorrect targets in target rotations, it is unclear which of these
46 potential errors is more serious. Given the self-correcting nature of target rotations mentioned
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3 previously, neither may be serious if a researcher allows an iterative method of defining a target
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5 rotation. Such iteration strategies are beyond the present scope, however.
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8 In what follows, we demonstrate the fitting of targeted rotations to multidimensional data
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10 in order to evaluate unidimensional and restricted bifactor IRT models. First, we evaluate the
11 ability of the SL to recover a good target matrix, and second, we evaluate the targeted rotations
12 themselves (assuming a correct target). In both cases, we specify a true population factor
13 loading matrix, and then we convert that matrix into a true population tetrachoric matrix Σ using
14 the relation given in Equation 13:
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$$\Sigma = \Lambda\Phi\Lambda' + \Psi. \quad (13)$$

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24 This tetrachoric matrix is then used to generate dichotomous data. A sample tetrachoric matrix is
25 estimated from the simulated dichotomous data and submitted to the analyses detailed in later
26 sections. In all analyses, we specified a structure with 15 items, and OFA threshold (or IRT
27 intercept) parameters are fixed to zero for all items. This was done to control for the possibility
28 that the estimation of loadings or slopes is affected by the value of the thresholds or intercepts,
29 respectively.
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38 **Simulations**

39 To explore the efficacy of the SL to suggest an initial target, we conducted two sets of
40 Monte Carlo analyses. In the first, we specified true population bifactor structures that were
41 independent cluster – each item loads on the general and one and only one group factor. For all
42 analyses, we specified 15 items with 5 items loading on each of three group factors. We defined
43 three levels of loadings: low = .3 to .5, medium = .4 to .6, and high = .5 to .7. The design was
44 then completely crossed with general and group factor loadings randomly selected from a
45 uniform distribution with boundaries set to either low, medium or high values (defined above).
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Target Rotations

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3 For example, conditions ran from general factor loadings high and group factor loadings low, to,
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5 general factor loadings low and group factor loadings high. Note that because we knew the true
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7 population pattern matrix, we automatically knew the correct target matrix. Note also that the
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9 true loadings specified in the leftmost columns of Tables 3-4 are means of the uniform
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11 distribution from which the simulated loadings were sampled. For example, loadings sampled
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13 from a uniform distribution ranging from 0.4 to 0.6 (“medium,” as defined above) would be
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15 indicated in the tables as 0.5 (the mean of said distribution).
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20 Based on the true loading matrices simulated above, we computed the implied population
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22 correlation matrix. This process was then repeated 1,000 times for three sample size conditions
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24 of 250, 500, and 1,000. For each simulated data set, we then estimated the tetrachoric correlation
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26 matrix, and conducted a SL orthogonalization extracting three group factors using unweighted
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28 least squares and rotating to an oblimin solution. For each SL result, we specified a target matrix
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30 using the .15 criterion mentioned above. The outcome of interest is simply the number of times
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32 each element of the target matrix was correctly specified.
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37 There are two types of mistakes possible: 1) a loading that is zero in the population is
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39 estimated above .15 and is thus left erroneously unspecified (?), and 2) a loading that is .30 and
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41 above in the population is underestimated in the data and gets erroneously specified (0). The
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43 results under IC conditions are very easy to summarize. When the data are IC cluster and sample
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45 size is 500 and above, errors in specifying the cells of the target matrix occur less than 1% of the
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47 time. Although we simulated a total of six conditions (strong general, strong groups; strong
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49 general, medium groups, etc.), each with sample sizes of 250, 500, and 1000, we show only the
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51 four most extreme conditions. Table 3 shows the percentages of times errors occurred in each
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53 cell of the target matrix under conditions of a strong general trait with strong group factors (top
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3 portion) and a weak general trait with weak group factors (bottom portion). Note that the
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5 proportion errors for the general factor are not shown, because those are always unspecified
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7 (correctly) in the comparison modeling procedure. As expected, strong general and group factor
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9 loadings are almost always discovered and specified correctly (for the target) by the SL, even in
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11 small sample sizes. In sample sizes above 500, it is fair to say the SL will always recover a
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13 correctly specified target in this strong general/strong groups condition.
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17 The condition involving a weak general factor combined with weak group factors (bottom
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19 portion of Table 3) is far more likely to cause problems for the SL when trying to recover the
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21 true target. This is because low loadings are more vulnerable to underestimation below 0.15,
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23 especially in small samples. Estimation below this threshold results in the target element
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25 erroneously being specified (0) rather than unspecified (?). Even given this problematic
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27 condition of weak general and group factors, however, sufficient sample size (500-1000) can
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29 correct the problem almost entirely.
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34 Table 4 shows the equivalent results for conditions in which the general factor is strong and
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36 the group factors are weak (top portion) and vice versa (bottom portion). Results in the top
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38 portion of Table 4 suggest that the existence of a strong general factor improves the SL's ability
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40 to recover the true target, but perhaps not enough to justify sample sizes below 500. Whereas a
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42 condition involving strong general *and* group factors (top portion of Table 3) appears robust to
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44 small samples, the same cannot necessarily be said of the condition shown in the top portion of
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46 Table 4. The bottom portion of Table 4 contains the results when the top portion is reversed.
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48 Interestingly, the existence of strong group factors (and a weak general factor) is even more
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50 conducive to correct target specification by the SL than the reverse condition (strong general,
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52 weak groups). Indeed, given strong group factors, one can expect the SL to specify a correct
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Target Rotations

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target ~90% of the time with a sample size of 250. Conditions involving medium general and group factor strengths are not shown, because these can be interpolated from the four extreme conditions in Tables 3 and 4.

Target Bifactor Rotations: How Accurate?

Given the results above, we now turn to consideration of the accuracy of targeted rotations under different conditions, assuming that the target matrix is correctly specified. As noted, rotations to a target structure are not new (e.g., Tucker, 1940), but the rotation of a factor pattern to a partially specified target matrix (Browne, 1972a, b; 2001) is only recently gaining increased attention due to the availability of software packages to implement targeted and other types of non-standard rotation methods (e.g., *MPLUS*; Asparouhov & Muthén, 2008; Comprehensive Exploratory Factor Analysis, *CEFA*; Browne, Cudeck, Tateneni, & Mels, 2004). The *MPLUS* program allows the user to conduct Monte Carlo simulations by specifying a true population loading matrix, and target pattern matrix of specified (0) and unspecified (?) elements. The WLS estimator was used.

“This target matrix reflects partial knowledge as to what the factor pattern should be” (Browne, 2001, p. 124). The target matrix forms the basis for a rotation that minimizes the sum of squared differences between the target and the rotated factor pattern. It is important to recognize that a specified element of a target pattern matrix is not the same as a fixed element in structural equation modeling. A fixed element in structural equation modeling is not allowed to vary at all from its specified value. If said value is unreasonable, it will manifest in poor fit statistics (CFI, RMSEA, etc.). In targeted rotations, however, a “fixed/specified” element is free to vary from its “specified” value if a different value will lead to better *overall* fit of the target to

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3 the estimated loading matrix. As mentioned above, in targeted rotation, “fit” is measured by the
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5 sum of squared deviations of the pattern matrix from the estimated matrix.
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8 Above, we focused on the first relevant question in evaluating the effectiveness of the
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10 comparison modeling approach: How often does the SL orthogonalization succeed in pinpointing
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12 a good target matrix? We now turn to the second question: Assuming the SL pinpoints a good
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14 target, how effective is the targeted rotation itself in recovering the population parameters?
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18 Whereas population parameters in the above simulations were sampled from a uniform
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20 distribution, population parameters in the present simulations were fixed at a single number.
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22 This was necessary to compare true and estimated parameters on a continuous scale, whereas in
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24 the previous simulations, we were interested only in counts of success/failure. The fixed
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26 population parameters used in the present simulations were specified simply as the means of the
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28 analogous uniform distribution from the previous simulations. For example, loadings drawn
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30 from a 0.3-0.5 uniform distribution in the above simulations are defined as 0.4 presently.
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34 The “Monte Carlo” procedure in MPLUS requires typical inputs: number of simulated
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36 samples, number of persons per sample, population parameters from which to generate
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38 observations, and the desired analysis to be run on said observations. Population parameters
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40 were specified as variations on bifactor independent cluster structure directly analogous to the
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42 structures simulated in the above Monte Carlo analyses. Weak/Moderate/Strong group and
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44 general factors were completely crossed with three sample sizes (250, 500, and 1000).
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48 Additionally, we simulated some extreme conditions, which are indicated below. The target
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50 matrix in each condition was specified perfectly—i.e., if a particular loading is zero in the
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52 population, it was specified as zero. Otherwise, MPLUS was told to estimate it.
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Target Rotations

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Table 5 (Demonstration A) presents true and estimated loadings when the general and group factors are equal and of moderate strength. To save space, only sample sizes of 250 and 1000 are shown. At a sample size of 250, loadings were recovered reasonably well, with an average absolute error of 0.020. Exceptions to this include item 4's loading on the first group factor, with an error of 0.08, and an apparent, small positive bias on the group factors. The larger sample size (1000) offered surprisingly little improvement. Average absolute error (AAE) decreased to 0.014, an improvement of only 0.006. Additionally, the group factors remained positively biased, and the worst error (item 6, group 3) was not much better than when $N = 250$.

Demonstration B (bottom portion of Table 5) presents true and estimated loadings when the general factor is strong but the group factors are weak. An immediately observable phenomenon is that the general factor (which is stronger than in Demonstration A) is estimated more accurately, with AAEs of 0.009 ($N = 250$) and 0.004 ($N = 1000$). This improved accuracy over Demonstration A extends to the group factors as well. Even though item 14's loading on group three is off by 0.10 when $N = 250$, the AAE was lower on the group factors in Demonstration A (despite their being weaker).

Table 6 contains two other independent cluster examples: a weak general factor with strong group factors (Demonstration C), and a weak general factor with unbalanced group factors (Demonstration D). Demonstration C is analogous to Demonstration B (Table 5), only reversed, such that the group factors actually account for more item variance than does the general factor. Interestingly, even though the communalities do not differ between Demonstrations B and C, the latter proved more challenging for the targeted rotation to estimate. At sample sizes of 250 and 1000, the AAE was 0.027 and 0.017, respectively (compared to 0.009 and 0.004 in Demonstration B). The source of this larger error in Demonstration C is due mainly to consistent

Target Rotations

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underestimation of the general factor loadings. While the strong general factor in Demonstration B was estimated almost perfectly by the targeted rotation, the weaker general factor in Demonstration C appears to be vulnerable to slight underestimation. As sample size increases, however, this underestimation approaches zero.

Demonstration D (also in Table 6) is the most complicated of the independent cluster examples shown presently, because the general factor is weak and the group factors vary from explaining 36% of item variance (G1) to only 16% (G3). It is also the most extreme example of sample size effects, with AAE dropping from 0.033 (N = 250) to 0.016 (N = 1000). This is due mainly to an apparent vulnerability of the small sample size to produce a few very large errors. Item 14's loading on G3 and item 4's loading on G1 are both misestimated by more than 0.10. Further simulations (not shown) confirmed that errors of this size are indeed common with this factor structure at small sample sizes. A second noticeable phenomenon in Demonstration D is a pattern of increasing underestimation of the general factor as the group factors decrease in size, with general factor loadings underestimated (on average) by 0.02 on G1, 0.04 on G2, and 0.06 on G3. This would indicate a misalignment of the general latent vector, obviously caused by the imbalanced group factors.

Table 7 displays the true and estimated factor loadings for more problematic conditions involving violations of independent cluster structure. In Demonstration E, the general and group factor loadings are moderate and equal, but item 3 cross-loads on the third group factor. Cross-loadings present problems for a few reasons, but one of the most important is that they indicate the existence of factors unaccounted for by the hypothesized structure. Discovery of cross-loadings is crucial to proper item parameter estimation, and should lead researchers to consider extracting additional factors or to remove the problematic item(s) altogether.

As in Demonstrations A-C, the effect of sample size in Demonstration E (Table 7) appears negligible, decreasing average absolute error from 0.024 ($N = 250$) to 0.021 ($N = 1000$). Additionally, even though we assume here that the target matrix is perfect, the targeted rotation still does not estimate the cross-loading (0.50, item 3) perfectly. Specifically, it tends to overestimate the general factor loading and underestimate both the group loading and the cross-loading. These over- and under-estimations, which average 0.09, are large enough to cause problems for item parameter estimation. Nonetheless, the Schmid-Leiman (not shown) performed even worse, over- and under-estimating the cross-loadings almost twice as badly. This lends support to our argument that performing a target rotation after performing a SL will lead to more accurate loading estimation, even though the target is not perfect. Further, with the exception of item 3, the target rotation recovered the true factor loadings with accuracy comparable to Demonstrations A through D.

Finally, the population factor structure in Demonstration F (also in Table 7) is bordering on incoherent. Given the severity of independent clusters (three unbalanced sets of two cross-loadings each), a researcher would be unlikely to overlook such a problem, and would probably deem the hypothesized factor structure unreasonable. We include Demonstration F here to test the limits of our proposed method. As expected, the average absolute error was greater for Demonstration F than for any of the previous demonstrations. The effect of sample size is also large, with AAE decreasing from 0.039 ($N = 250$) to 0.027 ($N = 1000$). Remarkably, however, given sufficient sample size (1000), the *largest* absolute error in Demonstration F is only 0.07. Indeed, while the SL orthogonalization failed miserably with such a complex structure (not shown), the targeted rotation recovered 90% of the loadings within 0.06 of their true values.

Discussion

Parameters can only be estimated correctly to the degree that the data are appropriate for a specific model. In an IRT context, when considering a highly restrictive unidimensional model, item response data must have a structure that is consistent with this restriction in order to accurately estimate population parameters. From an applied standpoint, accurate item parameter estimation is critical because it underlies all applications of IRT models including: a) scaling individual differences, b) performing scale linking, c) developing and administering computerized adaptive tests, d) analyzing the psychometric functioning of items and scales, and e) identifying differential item functioning.

Nevertheless, it is generally recognized that even well constructed psychological measures are unlikely to produce item response data that are perfectly consistent with the restrictions of a unidimensional model. In fact, it is arguable that many psychological measures produce item response matrices that are consistent with both a single general common factor (i.e., "strong" common factor) and multidimensional models caused by the presence of two or more content parcels. Precisely because the assessment of "unidimensionality" is so challenging, applied researchers rely heavily on rule-of-thumb guidelines to judge when a restricted model may be safely applied to a given dataset.

In this report, we reviewed and provided further empirical support for a proposed "comparison modeling" approach to evaluating the viability of particular restricted IRT model applications (Reise, Cook, & Moore, under review). Specifically, in comparison modeling, a researcher identifies an exploratory multidimensional factor rotation that is assumed to better represent the true structure of the data. In turn, a researcher then compares the item slopes with those estimated under a more restricted model. As stated by Browne (2001, p. 113) in the

context of structural equation modeling, “The discovery of misspecified loadings, however, is more direct through rotation of the factor matrix than through the examination of model modification indices.” We believe that a similar logic applies to IRT model fitting as well. That is, we believe that it is better to thoroughly understand the multidimensional structure of the data, and to identify potential modeling problems, prior to fitting an IRT model as opposed to either relying on a rule-of-thumb (e.g., ratio of 1st to 2nd eigenvalue greater than 3), or fitting an IRT model and then searching for inadequacies (e.g., inspection of correlated residuals or fit indices).

As illustrated by the present research, at their best (an adequate sample size and a multidimensional structure that is independent cluster), a Schmid-Leiman orthogonalization can yield an adequate initial target pattern. In turn, assuming the target is correct, a target rotation can provide, in either a factor analytic or IRT metric, an accurate unrestricted bifactor comparison model. Finally, this bifactor comparison model can provide a direct index of the degree of item parameter bias caused by forcing multidimensional data into unidimensional models. Of course, we note that no targeted rotation procedure can be expected to work under all the possible conditions that confront a researcher. Moreover, the application of targeted rotations is much more complicated when working with real data, as opposed to the hypothetical examples provided here. Thus, in the next section, we briefly consider challenges and limitations of targeted rotations and describe directions for future research.

Limitations and Future Directions

The most significant challenge of comparison IRT modeling is to identify a correct exploratory factor model to compare with a more restricted model. The ability to accomplish this goal depends on: a) the true structure of the data (e.g., the severity and form of multidimensionality), b) the ability of the target rotation method to recover this true structure

Target Rotations

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3 (e.g., what particular function should be minimized?; should weighted or unweighted least
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5 squares be used?), which depends partially on, c) how the researcher specifies an initial target
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7 pattern (e.g., based on theory or a priori data analysis?). We will focus the following discussion
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9 on this latter task – identifying the dimensionality of a target matrix and specifying the values of
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11 an initial target.
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15 To specify a target matrix, the first challenge is the historically important problem of
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17 identifying the dimensionality of the target matrix. Recall that in order to test a restricted model
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19 (e.g., unidimensionality), the researcher needs to compare that with an exploratory
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21 multidimensional model. Drawing partly from experience and partly from the present results, we
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23 suggest that in many situations, an unweighted least squares extraction followed by an oblique
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25 rotation can provide a lot of insight into the questions of how many factors should be extracted
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27 as well as the question of how the target should be specified. As shown in the above simulations,
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29 when the data are multidimensional due to having at least three "clusters" of items, and the items
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31 have an independent cluster structure, the SL is capable of adequately recovering the
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33 configuration of meaningful loadings. However, although not explicitly illustrated by the present
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35 simulations, it is easy to demonstrate that the SL can run into serious problems when the data
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37 depart drastically from independent cluster structure (i.e., many cross-loadings in an oblique
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39 rotated solution). On the other hand, the degree to which any model should be applied to such
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41 data is highly questionable.
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49 In closing, we note that although it is critical to correctly specify the dimensionality of
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51 the initial target matrix, it may not be so critical to correctly specify the particular pattern of
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53 specified and unspecified elements in that matrix. We have observed that if a wrong element is
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55 specified (e.g., a zero is placed for a loading that is truly non-zero), MPLUS will identify this
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3 problem by estimating a non-zero rotated factor loading. In turn, the researcher may then re-
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5 specify the target matrix, and re-estimate the rotation. The opposite is also true in many
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7 circumstances. That is, if a researcher mistakenly specified a question mark for an element that
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9 should be zero, MPLUS can identify this problem. The conditions under which a researcher can
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11 iterate to the correct target in this manner remains to be demonstrated in future studies.
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15 Summary

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17 We described the use of exploratory bifactor IRT models (estimated as exploratory factor
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19 models) to judge the effects of forcing multidimensional data into a unidimensional model. We
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21 propose that the comparison modeling approach allows researchers to evaluate the item
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23 parameter bias caused by forcing data into a restricted model. At the very least, exploratory
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25 modeling forces the researcher into considering alternative multidimensional models and thus to
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27 seriously considering the structure of their data and its impact on item parameter estimates. For
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29 this reason, we suggest that research that proposes a unidimensional IRT application should be
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31 required to also report the tetrachoric or polychoric correlation matrix so that other researchers
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33 can make their own judgments.
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Target Rotations

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TABLE 1
 Demonstration of OFA and IRT Transformations

Item	<u>Unidimensional</u>		<u>Multidimensional</u>							
	<u>FA</u>	<u>IRT</u>	λ_g	<u>FA</u>			<u>IRT</u>			
	λ	α		λ_1	λ_2	λ_3	α_g	α_1	α_2	α_3
1	0.78	1.17	0.60	0.50			0.96	0.80		
2	0.78	1.17	0.60	0.50			0.96	0.80		
3	0.78	1.17	0.60	0.50			0.96	0.80		
4	0.78	1.17	0.60	0.50			0.96	0.80		
5	0.78	1.17	0.60	0.50			0.96	0.80		
6	0.67	0.90	0.60		0.30		0.81		0.40	
7	0.67	0.90	0.60		0.30		0.81		0.40	
8	0.67	0.90	0.60		0.30		0.81		0.40	
9	0.67	0.90	0.60		0.30		0.81		0.40	
10	0.67	0.90	0.60		0.30		0.81		0.40	
11	0.61	0.77	0.60			0.10	0.76			0.13
12	0.61	0.77	0.60			0.10	0.76			0.13
13	0.61	0.77	0.60			0.10	0.76			0.13
14	0.61	0.77	0.60			0.10	0.76			0.13
15	0.61	0.77	0.60			0.10	0.76			0.13
Item	<u>Unidimensional</u>		<u>Multidimensional</u>							
	<u>IRT</u>	<u>FA</u>	α_g	<u>IRT</u>			<u>FA</u>			
	λ	α		α_1	α_2	α_3	λ_g	λ_1	λ_2	λ_3
1	1.17	0.78	0.75	0.58			0.54	0.42		
2	1.17	0.78	0.75	0.58			0.54	0.42		
3	1.17	0.78	0.75	0.58			0.54	0.42		
4	1.17	0.78	0.75	0.58			0.54	0.42		
5	1.17	0.78	0.75	0.58			0.54	0.42		
6	0.90	0.67	0.75		0.32		0.58		0.25	
7	0.90	0.67	0.75		0.32		0.58		0.25	
8	0.90	0.67	0.75		0.32		0.58		0.25	
9	0.90	0.67	0.75		0.32		0.58		0.25	
10	0.90	0.61	0.75		0.32		0.58		0.25	
11	0.77	0.61	0.75			0.10	0.60			0.08
12	0.77	0.61	0.75			0.10	0.60			0.08
13	0.77	0.61	0.75			0.10	0.60			0.08
14	0.77	0.61	0.75			0.10	0.60			0.08
15	0.77	0.61	0.75			0.10	0.60			0.08

TABLE 2
The Schmid-Leiman Orthogonalization Under Three Conditions

<i>True Population Structure</i>												
<i>IC: Proportional</i>					<i>IC: Not Proportional</i>				<i>IC Basis</i>			
<i>Item</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
1	.50	.60			.40	.27			.50	.60	.50	
2	.50	.60			.40	.68			.50	.60		
3	.50	.60			.40	.65			.50	.60		
4	.50	.60			.40	.21			.50	.60		
5	.50	.60			.40	.65			.50	.60		
6	.50		.50		.40		.43		.50		.50	.50
7	.50		.50		.40		.51		.50		.50	
8	.50		.50		.40		.29		.50		.50	
9	.50		.50		.40		.27		.50		.50	
10	.50		.50		.40		.19		.50		.50	
11	.50			.40	.40			.61	.50	.50		.40
12	.50			.40	.40			.13	.50			.40
13	.50			.40	.40			.46	.50			.40
14	.50			.40	.40			.64	.50			.40
15	.50			.40	.40			.53	.50			.40
<i>Schmid-Leiman</i>												
<i>Item</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
1	.50	.60	.00	.00	.33	.32	.09	.05	.65	.49	.36	.11
2	.50	.60	.00	.00	.41	.67	.01	.01	.49	.61	.01	.02
3	.50	.60	.00	.00	.40	.65	.00	.00	.49	.61	.01	.02
4	.50	.60	.00	.00	.31	.27	.10	.06	.49	.61	.01	.02
5	.50	.60	.00	.00	.40	.65	.00	.00	.49	.61	.01	.02
6	.50	.00	.50	.00	.41	.01	.41	.01	.61	.05	.40	.38
7	.50	.00	.50	.00	.43	.02	.45	.02	.50	.01	.50	.03
8	.50	.00	.50	.00	.37	.02	.33	.02	.50	.01	.50	.03
9	.50	.00	.50	.00	.36	.02	.32	.03	.50	.01	.50	.03
10	.50	.00	.50	.00	.33	.04	.27	.05	.50	.01	.50	.03
11	.50	.00	.00	.40	.41	.01	.01	.60	.56	.45	.05	.33
12	.50	.00	.00	.40	.30	.07	.12	.20	.42	.05	.04	.48
13	.50	.00	.00	.40	.38	.02	.03	.48	.42	.05	.04	.48
14	.50	.00	.00	.40	.42	.01	.02	.63	.42	.05	.04	.48
15	.50	.00	.00	.40	.39	.01	.01	.54	.42	.05	.04	.48

Target Rotations

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TABLE 3
Effects of Sample Size on the Ability of the SL to Specify a Correct Target

<i>Demonstration A: Strong General, Strong Groups</i>												
<i>Frequency of Incorrect Specification (%)</i>												
<i>True Loadings</i>				<i>N = 250</i>			<i>N = 500</i>			<i>N = 1000</i>		
<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
0.60	0.60			0	3	1	0	0	0	0	0	0
0.60	0.60			0	0	2	0	0	0	0	0	0
0.60	0.60			0	1	2	0	0	0	0	0	0
0.60	0.60			0	1	1	0	0	0	0	0	0
0.60	0.60			0	3	2	0	0	0	0	0	0
0.60		0.60		2	0	1	0	0	1	0	0	0
0.60		0.60		1	0	2	0	0	0	0	0	0
0.60		0.60		1	0	0	0	0	0	0	0	0
0.60		0.60		3	0	3	0	0	0	0	0	0
0.60		0.60		1	0	2	0	0	0	0	0	0
0.60			0.60	2	0	0	1	0	0	0	0	0
0.60			0.60	0	3	0	0	0	0	0	0	0
0.60			0.60	1	1	0	0	0	0	0	0	0
0.60			0.60	1	1	0	0	0	0	0	0	0
0.60			0.60	4	4	0	0	0	0	0	0	0
<i>Demonstration B: Weak General, Weak Groups</i>												
<i>Frequency of Incorrect Specification (%)</i>												
<i>True Loadings</i>				<i>N = 250</i>			<i>N = 500</i>			<i>N = 1000</i>		
<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
0.40	0.40			3	17	15	0	4	2	0	0	0
0.40	0.40			1	16	18	1	4	1	0	1	1
0.40	0.40			2	24	14	2	5	3	0	0	0
0.40	0.40			2	22	13	1	3	2	0	0	2
0.40	0.40			1	18	17	2	10	6	0	3	0
0.40		0.40		25	6	15	6	0	5	0	0	1
0.40		0.40		16	4	19	2	0	3	0	0	0
0.40		0.40		20	5	17	4	1	6	1	0	2
0.40		0.40		19	2	9	4	0	6	1	0	0
0.40		0.40		14	2	21	6	0	9	0	0	0
0.40			0.40	23	12	1	5	5	0	0	0	0
0.40			0.40	23	10	1	3	6	0	0	3	0
0.40			0.40	19	15	0	2	4	0	2	1	0
0.40			0.40	13	15	2	7	5	1	0	1	0
0.40			0.40	20	24	3	6	1	0	1	0	0

Target Rotations

TABLE 4
Effects of Sample Size on the Ability of the SL to Specify a Correct Target

<i>Demonstration C: Strong General, Weak Groups</i>												
<i>Frequency of Incorrect Specification (%)</i>												
<i>True Loadings</i>				<i>N = 250</i>			<i>N = 500</i>			<i>N = 1000</i>		
<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
0.60	0.40			0	13	7	0	4	4	0	0	0
0.60	0.40			2	13	20	0	4	2	0	0	0
0.60	0.40			0	13	9	0	4	4	0	0	0
0.60	0.40			0	15	9	0	2	0	0	0	0
0.60	0.40			0	4	11	0	2	4	0	0	0
0.60		0.40		7	0	13	0	0	4	0	0	0
0.60		0.40		20	0	7	0	0	2	0	0	0
0.60		0.40		15	0	7	4	0	2	0	0	0
0.60		0.40		17	0	11	2	0	4	0	0	2
0.60		0.40		11	2	7	2	0	7	0	0	0
0.60			0.40	13	15	0	2	2	0	2	0	0
0.60			0.40	17	13	0	2	0	0	0	0	0
0.60			0.40	9	22	0	0	2	0	0	0	0
0.60			0.40	22	15	0	2	2	0	0	0	0
0.60			0.40	13	13	0	0	2	0	0	0	0
<i>Demonstration D: Weak General, Strong Groups</i>												
<i>Frequency of Incorrect Specification (%)</i>												
<i>True Loadings</i>				<i>N = 250</i>			<i>N = 500</i>			<i>N = 1000</i>		
<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
0.40	0.60			2	4	4	0	0	0	0	0	0
0.40	0.60			2	4	7	0	0	0	0	0	0
0.40	0.60			2	2	2	0	0	0	0	0	0
0.40	0.60			2	4	9	0	0	0	0	1	0
0.40	0.60			2	9	2	0	0	2	0	0	0
0.40		0.60		4	0	2	0	0	0	0	0	0
0.40		0.60		2	0	7	0	0	2	0	0	0
0.40		0.60		7	0	4	0	0	0	0	0	0
0.40		0.60		2	0	2	0	0	2	0	0	0
0.40		0.60		2	0	7	2	0	0	0	0	0
0.40			0.60	4	4	0	0	0	0	0	0	0
0.40			0.60	2	4	0	0	0	0	1	0	0
0.40			0.60	4	7	0	0	0	0	0	0	0
0.40			0.60	2	9	0	0	5	0	0	0	0
0.40			0.60	2	11	0	0	0	0	0	0	0

TABLE 5
True and Estimated Loadings by Monte Carlo Simulation in Mplus

Demonstration A: Simple Structure (Moderate General)

Item	Population (True)				Estimated, N = 250				Estimated, N = 1000			
	Gen	G1	G2	G3	Gen	G1	G2	G3	Gen	G1	G2	G3
1	0.50	0.50			0.48	0.51	0.01	0.02	0.50	0.49	-0.01	-0.01
2	0.50	0.50			0.47	0.51	0.04	0.02	0.46	0.52	0.02	0.02
3	0.50	0.50			0.50	0.49	-0.01	0.00	0.48	0.54	0.00	0.01
4	0.50	0.50			0.47	0.58	0.02	0.02	0.48	0.51	0.01	0.02
5	0.50	0.50			0.49	0.50	0.01	0.02	0.47	0.55	0.01	0.02
6	0.50		0.50		0.48	0.02	0.50	0.03	0.50	0.01	0.51	0.01
7	0.50		0.50		0.49	0.01	0.51	0.00	0.52	0.00	0.47	0.00
8	0.50		0.50		0.47	0.02	0.53	0.02	0.52	-0.01	0.51	0.00
9	0.50		0.50		0.48	0.02	0.52	0.02	0.51	0.01	0.50	0.01
10	0.50		0.50		0.50	0.01	0.48	0.01	0.50	0.02	0.51	0.01
11	0.50			0.50	0.46	0.02	0.01	0.53	0.48	0.01	0.02	0.50
12	0.50			0.50	0.45	0.02	0.03	0.55	0.48	0.02	0.01	0.56
13	0.50			0.50	0.47	0.03	0.04	0.54	0.49	0.01	0.00	0.52
14	0.50			0.50	0.50	0.01	-0.01	0.49	0.48	0.01	0.01	0.52
15	0.50			0.50	0.47	0.01	0.02	0.52	0.49	0.01	0.01	0.50

Demonstration B: Simple Structure (Strong General, Weak Groups)

Item	Population (True)				Estimated, N = 250				Estimated, N = 1000			
	Gen	G1	G2	G3	Gen	G1	G2	G3	Gen	G1	G2	G3
1	0.60	0.40			0.60	0.43	0.02	0.02	0.60	0.39	0.00	-0.01
2	0.60	0.40			0.60	0.40	0.01	-0.01	0.60	0.39	0.00	0.00
3	0.60	0.40			0.60	0.40	0.01	0.02	0.59	0.41	0.02	0.01
4	0.60	0.40			0.59	0.46	0.00	0.01	0.59	0.44	0.01	0.01
5	0.60	0.40			0.60	0.37	0.00	0.01	0.59	0.42	0.00	-0.01
6	0.60		0.40		0.59	0.02	0.40	0.01	0.60	0.00	0.39	-0.01
7	0.60		0.40		0.59	-0.01	0.41	0.00	0.59	0.01	0.40	0.01
8	0.60		0.40		0.59	0.00	0.48	0.02	0.60	0.01	0.40	0.02
9	0.60		0.40		0.59	0.01	0.42	0.01	0.59	0.01	0.41	0.01
10	0.60		0.40		0.59	0.02	0.42	0.02	0.60	0.02	0.41	0.00
11	0.60			0.40	0.59	0.00	0.00	0.40	0.60	0.01	0.01	0.43
12	0.60			0.40	0.58	0.02	0.02	0.44	0.60	0.00	0.00	0.40
13	0.60			0.40	0.60	-0.02	0.01	0.39	0.60	0.01	0.00	0.40
14	0.60			0.40	0.58	0.03	0.02	0.50	0.60	0.00	0.01	0.39
15	0.60			0.40	0.59	0.01	0.00	0.44	0.60	-0.01	0.01	0.42

TABLE 6
True and Estimated Loadings by Monte Carlo Simulation in Mplus

Demonstration C: Simple Structure (Weak General, Strong Groups)

Item	<i>Population (True)</i>				<i>Estimated, N = 250</i>				<i>Estimated, N = 1000</i>			
	Gen	G1	G2	G3	Gen	G1	G2	G3	Gen	G1	G2	G3
1	0.40	0.60			0.40	0.60	0.01	0.00	0.36	0.64	0.02	0.02
2	0.40	0.60			0.36	0.61	0.04	0.01	0.38	0.59	0.00	0.01
3	0.40	0.60			0.37	0.63	0.02	0.02	0.37	0.64	0.01	0.01
4	0.40	0.60			0.36	0.64	0.02	0.03	0.36	0.60	0.02	0.02
5	0.40	0.60			0.38	0.58	0.03	0.03	0.38	0.66	0.01	0.01
6	0.40		0.60		0.36	0.02	0.65	0.03	0.39	0.01	0.64	0.01
7	0.40		0.60		0.34	0.02	0.60	0.02	0.39	0.01	0.58	0.01
8	0.40		0.60		0.35	0.02	0.61	0.03	0.39	0.01	0.64	0.02
9	0.40		0.60		0.36	0.02	0.60	0.02	0.40	0.01	0.58	0.01
10	0.40		0.60		0.34	0.04	0.62	0.03	0.39	0.02	0.58	0.01
11	0.40			0.60	0.36	0.02	0.02	0.61	0.40	0.01	0.00	0.56
12	0.40			0.60	0.37	0.01	0.02	0.62	0.39	0.01	0.01	0.59
13	0.40			0.60	0.37	0.02	0.03	0.60	0.38	0.02	0.02	0.60
14	0.40			0.60	0.35	0.03	0.02	0.65	0.38	0.02	0.02	0.63
15	0.40			0.60	0.35	0.02	0.03	0.62	0.40	0.00	0.01	0.63

Demonstration D: Simple Structure (Moderate General, Unbalanced Groups)

Item	<i>Population (True)</i>				<i>Estimated, N = 250</i>				<i>Estimated, N = 1000</i>			
	Gen	G1	G2	G3	Gen	G1	G2	G3	Gen	G1	G2	G3
1	0.40	0.60			0.44	0.55	-0.02	-0.01	0.38	0.60	0.00	0.00
2	0.40	0.60			0.36	0.62	0.03	0.03	0.35	0.62	0.02	0.02
3	0.40	0.60			0.37	0.62	0.02	0.03	0.38	0.61	0.00	0.00
4	0.40	0.60			0.36	0.71	0.02	0.03	0.36	0.62	0.02	0.02
5	0.40	0.60			0.39	0.57	0.02	0.02	0.35	0.64	0.02	0.02
6	0.40		0.50		0.39	0.01	0.50	0.03	0.39	0.01	0.49	0.01
7	0.40		0.50		0.37	0.01	0.53	0.02	0.41	0.01	0.53	0.00
8	0.40		0.50		0.35	0.02	0.51	0.03	0.40	0.01	0.47	0.02
9	0.40		0.50		0.36	0.03	0.51	0.03	0.40	0.01	0.51	0.02
10	0.40		0.50		0.36	0.02	0.52	0.02	0.42	0.01	0.50	-0.01
11	0.40			0.40	0.36	0.01	0.03	0.40	0.38	0.02	0.02	0.46
12	0.40			0.40	0.34	0.02	0.03	0.47	0.39	0.01	0.00	0.40
13	0.40			0.40	0.33	0.03	0.06	0.47	0.38	0.03	0.02	0.46
14	0.40			0.40	0.34	0.04	0.02	0.52	0.40	0.01	0.00	0.42
15	0.40			0.40	0.34	0.03	0.03	0.43	0.38	0.01	0.02	0.38

TABLE 7
True and Estimated Loadings by Monte Carlo Simulation in Mplus

Demonstration E: IC Basis, Moderate General, Balanced Groups, One Cross-Loading

Item	<i>Population (True)</i>				<i>Estimated, N = 250</i>				<i>Estimated, N = 1000</i>			
	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
1	0.50	0.50			0.49	0.53	0.03	0.01	0.49	0.52	0.01	0.00
2	0.50	0.50			0.47	0.54	0.04	0.02	0.47	0.51	0.03	0.02
3	0.50	0.50		0.50	0.58	0.40	-0.06	0.42	0.60	0.43	-0.08	0.37
4	0.50	0.50			0.46	0.54	0.03	0.04	0.47	0.51	0.03	0.02
5	0.50	0.50			0.47	0.51	0.04	0.02	0.47	0.55	0.02	0.01
6	0.50		0.50		0.48	0.01	0.51	0.00	0.47	0.01	0.57	0.02
7	0.50		0.50		0.47	0.00	0.51	0.00	0.49	0.00	0.51	-0.01
8	0.50		0.50		0.46	0.03	0.54	0.02	0.49	0.00	0.50	0.01
9	0.50		0.50		0.46	0.02	0.53	0.04	0.48	0.01	0.50	0.01
10	0.50		0.50		0.49	0.02	0.48	0.01	0.49	0.02	0.50	0.00
11	0.50			0.50	0.47	0.03	0.05	0.48	0.49	0.01	0.02	0.51
12	0.50			0.50	0.50	0.00	0.01	0.50	0.49	0.01	0.02	0.54
13	0.50			0.50	0.52	0.01	0.01	0.48	0.50	0.01	0.02	0.48
14	0.50			0.50	0.49	0.03	0.02	0.52	0.50	0.00	0.01	0.54
15	0.50			0.50	0.50	0.01	0.01	0.50	0.50	0.01	0.02	0.49

Demonstration F: IC Basis, Weak General, Unbalanced Groups, Six Unbalanced Cross-Loadings

Item	<i>Population (True)</i>				<i>Estimated, N = 250</i>				<i>Estimated, N = 1000</i>			
	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>	<i>Gen</i>	<i>G1</i>	<i>G2</i>	<i>G3</i>
1	0.40	0.60		0.20	0.45	0.55	0.01	0.14	0.42	0.59	0.00	0.16
2	0.40	0.60		0.20	0.40	0.56	0.02	0.17	0.40	0.60	0.01	0.18
3	0.40	0.60			0.36	0.64	0.03	0.02	0.37	0.62	0.03	0.00
4	0.40	0.60			0.35	0.66	0.02	0.00	0.35	0.63	0.03	0.01
5	0.40	0.60			0.39	0.58	0.02	0.02	0.36	0.61	0.02	0.01
6	0.40		0.50		0.32	0.04	0.58	0.02	0.33	0.03	0.53	0.02
7	0.40		0.50		0.34	0.01	0.53	0.02	0.35	0.02	0.51	0.02
8	0.40		0.50	0.40	0.43	-0.02	0.45	0.35	0.44	-0.01	0.48	0.35
9	0.40		0.50	0.40	0.43	0.00	0.46	0.33	0.45	-0.01	0.48	0.33
10	0.40		0.50		0.30	0.05	0.63	0.03	0.36	0.02	0.56	0.00
11	0.40			0.40	0.36	0.02	0.03	0.41	0.40	0.01	0.03	0.37
12	0.40			0.40	0.35	0.03	0.04	0.45	0.41	0.01	0.01	0.38
13	0.40			0.40	0.37	0.03	0.04	0.41	0.40	0.01	0.02	0.38
14	0.40	0.60		0.40	0.45	0.55	-0.02	0.32	0.46	0.57	-0.02	0.33
15	0.40	0.60		0.40	0.46	0.57	-0.03	0.32	0.46	0.56	-0.02	0.33