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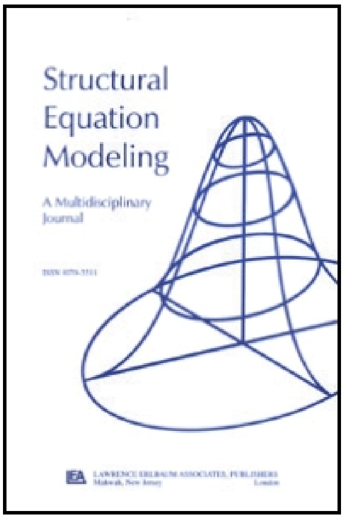

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Exploratory Structural Equation Modeling

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Exploratory factor analysis (EFA) is a frequently used multivariate analysis technique in statistics. Jennrich and Sampson (1966) solved a significant EFA factor loading matrix rotation problem by deriving the direct Quartimin rotation. Jennrich was also the first to develop standard errors for rotated solutions, although these have still not made their way into most statistical software programs. This is perhaps because Jennrich's achievements were partly overshadowed by the subsequent development of confirmatory factor analysis (CFA) by Jöreskog (1969). The strict requirement of zero cross-loadings in CFA, however, often does not fit the data well and has led to a tendency to rely on extensive model modification to find a well-fitting model. In such cases, searching for a well-fitting measurement model may be better carried out by EFA (Browne, 2001). Furthermore, misspecification of zero loadings usually leads to distorted factors with over-estimated factor correlations and subsequent distorted structural relations. This article describes an EFA-SEM (ESEM) approach, where in addition to or instead of a CFA measurement model, an EFA measurement model with rotations can be used in a structural equation model. The ESEM approach has recently been implemented in the M*plus* program. ESEM gives access to all the usual SEM parameters and the loading rotation gives a transformation of structural coefficients as well. Standard errors and overall tests of model fit are obtained. Geomin and Target rotations are discussed. Examples of ESEM models include multiple-group EFA with measurement and structural invariance testing, test–retest (longitudinal) EFA, EFA with covariates and direct effects, and EFA with correlated residuals. Testing strategies with sequences of EFA and CFA models are discussed. Simulated and real data are used to illustrate the points.

The latent variable measurement specification in structural equation modeling (SEM; Bollen, 1989; Browne & Arminger, 1995; Jöreskog & Sorbom, 1979; Muthén, 1984) uses the Jöreskog (1969) confirmatory factor analysis (CFA) model. Based on theory and prior analyses, the CFA measurement model specifies a number of factor loadings fixed at zero to reflect a hypothesis that only certain factors influence certain factor indicators. Often a simple structure is specified where each indicator is influenced by a single factor; that is, there are no cross-

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loadings, sometimes referred to as variable complexity of one. The number of such zero loading restrictions is typically much larger than the number of restrictions needed to identify the factor analysis measurement model, which as in exploratory factor analysis (EFA) with m factors is m^2 restrictions on the factor loadings, factor variances, and factor covariances. The use of CFA measurement modeling in SEM has the advantage that researchers are encouraged to formalize their measurement hypotheses and develop measurement instruments that have a simple measurement structure. Incorporating a priori substantive knowledge in the form of restrictions on the measurement model makes the definition of the latent variables better grounded in subject-matter theory and leads to parsimonious models.

The use of CFA measurement modeling in SEM also has disadvantages and these are likely to have contributed to poor applications of SEM where the believability and replicability of the final model is in doubt. Although technically appealing, CFA requires strong measurement science that is often not available in practice. A measurement instrument often has many small cross-loadings that are well motivated by either substantive theory or by the formulation of the measurements. The CFA approach of fixing many or all cross-loadings at zero may therefore force a researcher to specify a more parsimonious model than is suitable for the data. Because of this, models often do not fit the data well and there is a tendency to rely on extensive model modification to find a well-fitting model. Here, searching for a well-fitting measurement model is often aided by the use of model modification indexes. A critique of the use of model searches using modification indexes is given, for example, in MacCallum, Roznowski, and Necowitz (1992). In such situations of model uncertainty, Browne (2001) advocates exploratory rather than confirmatory approaches:

> Confirmatory factor analysis procedures are often used for exploratory purposes. Frequently a confirmatory factor analysis, with pre-specified loadings, is rejected and a sequence of modifications of the model is carried out in an attempt to improve fit. The procedure then becomes exploratory rather than confirmatory.—In this situation the use of exploratory factor analysis, with rotation of the factor matrix, appears preferable.—The discovery of misspecified loadings ... is more direct through rotation of the factor matrix than through the examination of model modification indices. (p. 113)

Furthermore, misspecification of zero loadings in CFA tends to give distorted factors. When nonzero cross-loadings are specified as zero, the correlation between factor indicators representing different factors is forced to go through their main factors only, usually leading to overestimated factor correlations and subsequent distorted structural relations.

For the reasons just given, it is important to extend SEM to allow less restrictive measurement models to be used in tandem with the traditional CFA models. This offers a richer set of a priori model alternatives that can be subjected to a testing sequence. This article describes an exploratory structural equation modeling (ESEM) approach, where in addition to or instead of CFA measurement model parts, EFA measurement model parts with factor loading matrix rotations can be used. For each EFA measurement model part with m factors, only m^2 restrictions are imposed on the factor loading matrix and the factor covariance matrix. ESEM gives access to all the usual SEM parameters, such as residual correlations, regressions of factors on covariates, and regressions among factors. Multiple-group analysis with intercept and mean structures are also handled. The ESEM approach has recently been implemented in the M*plus* program.

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EFA is a frequently used multivariate analysis technique in statistics. Jennrich and Sampson (1966) solved a significant EFA factor loading matrix rotation problem by deriving the direct Quartimin rotation. Jennrich was also the first to develop standard errors for rotated solutions. Cudeck and O'Dell (1994) provided a useful discussion on the benefits of considering standard errors for the rotated factor loadings and factor correlation matrix in EFA. However, EFA standard errors have still not made their way into most statistical software programs (Jennrich, 2007), perhaps because Jennrich's achievements were partly overshadowed by the subsequent development of CFA by Jöreskog (1969). The work to be presented can therefore also be seen as a further development and modernization of EFA, continuing the classic psychometric work that was largely abandoned. Three examples can be mentioned. Correlated residuals among factor indicators sharing similar wording can confound the detection of more meaningful factors using conventional EFA; allowing such parameters in an extended EFA can now give new measurement insights. Comparing EFA factor loadings across time in longitudinal studies or across groups of individuals can now be done using rigorous likelihood-ratio testing without the researcher being forced to switch from EFA to CFA.

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It should be made clear that the development in this article is not intended to encourage a complete replacement of CFA with EFA measurement modeling in SEM. Instead, the intention is to add further modeling flexibility by providing an option that in some cases is more closely aligned with reality, reflecting more limited measurement knowledge of the researcher or a more complex measurement structure. There will still be many situations where a CFA approach is preferred. Apart from situations where the measurement instruments are well understood, this includes applications where a CFA specification lends necessary stability to the modeling. As one example, multitrait, multimethod (MTMM) modeling relies on CFA specification of both the trait and the methods part of the model. Although it is in principle possible with the methods presented here to let the trait part be specified via EFA, leaving the methods part specified as CFA, this might not provide easy recovery of the data-generating parameter values.

In ESEM, the loading matrix rotation gives a transformation of both measurement and structural coefficients. Extending the work summarized in Jennrich (2007), ESEM provides standard errors for all rotated parameters. Overall tests of model fit are also obtained. With EFA measurement modeling, the reliance on a good rotation method becomes important. This article discusses the Geomin rotation (Yates, 1987), which is advantageous with variable complexity greater than one (Browne, 2001; McDonald, 2005). Target rotation (Browne, 2001) is a less-known rotation technique that conceptually is situated in between EFA and CFA, which is also implemented in the general ESEM framework. Examples of ESEM models are presented including multiple-group EFA with measurement invariance.[1] Testing strategies with sequences of EFA and CFA models are discussed. Simulated and real data are used to illustrate the points.

The outline of this article is as follows. First a simple ESEM model is presented. Next the general ESEM model is described as well as an outline of the estimation method. The ESEM modeling is then expanded to include constrained rotation methods that are used to estimate, for example, measurement-invariant ESEM models and multiple-group EFA models. Various rotation criteria and their properties are described after that. An empirical example is

[1]Examples of ESEM models illustrating structural invariance testing, EFA with covariates and direct effects, and EFA with correlated residuals are available in Bengt Muthén's multimedia presentation on this topic available at http://www.ats.ucla.edu/stat/mplus/seminars/whatsnew_in_mplus5_1/default.htm

also presented to illustrate the advantages of ESEM in real-data modeling. Several simulation studies are then presented. The choice of the rotation criterion is also discussed. The article concludes with a summary of the presented methodology.

SIMPLE EXPLORATORY STRUCTURAL EQUATION MODEL

Suppose that there are p dependent variables $Y = (Y_1, \ldots, Y_p)$ and q independent variables $X = (X_1, \ldots, X_q)$. Consider the general structural equation model with m latent variables $\eta = (\eta_1, \ldots, \eta_m)$

$$Y = \nu + \Lambda\eta + KX + \varepsilon \tag{1}$$

$$\eta = \alpha + B\eta + \Gamma X + \zeta \tag{2}$$

The standard assumptions of this model are that the ε and ζ are normally distributed residuals with mean 0 and variance covariance matrix Θ and Ψ, respectively. The model can be extended to multiple-group analysis, where for each group model (1-2) is estimated and some of the model parameters can be the same in the different groups. The model can also be extended to include categorical variables and censored variables as in Muthén (1984) using limited-information weighted least squares estimation. For each categorical and censored variable Y^* is used instead of Y in Equation 1, where Y^* is an underlying unobserved normal variable. For each categorical variable there is a set of parameters τ_k such that

$$Y = k \Leftrightarrow \tau_k < Y^* < \tau_{k+1}. \tag{3}$$

Thus a linear regression for Y^* is equivalent to a Probit regression for Y. Similarly, for censored variables with a censoring limit of c

$$Y = \begin{cases} Y^* & \text{if } Y \geq c \\ c & \text{if } Y \leq c \end{cases} \tag{4}$$

All of the parameters in the preceding model can be estimated with the maximum likelihood estimation method, however, this structural model is generally unidentified and typically many restrictions need to be imposed on the model. Otherwise the maximum likelihood estimates will be simply one set of parameter estimates among many equivalent solutions.

One unidentified component is the scale of the latent variable. Two different approaches are generally used to resolve this nonidentification. The first approach is to identify the scale of the latent variable by fixing its variance to 1. The second approach is to fix one of the Λ parameters in each column to 1. The two approaches are generally equivalent and a simple reparameterization can be used to obtain the parameter estimates from one to the other scales. In what follows the first approach is taken. It is assumed that the variance of each latent variable is 1. Later on the model is expanded to include latent factors with scale identified by the second method. It is also assumed in this section that all Λ parameters are estimated.

Even when the scale of the latent variable is identified, however, there are additional identifiability issues when the number of latent factors m is greater than 1. For each square

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matrix H of dimension m one can replace the η vector by $H\eta$ in model (1-2). The parameters in the model will be altered as well. The Λ will be replaced by ΛH^{-1}, the α vector is replaced by $H\alpha$, the Γ matrix is replaced by $H\Gamma$, the B matrix is replaced by HBH^{-1} and the Ψ matrix is replaced by $H\Psi H^T$. Because H has m^2 elements the model has a total of m^2 indeterminacies. In the discussion that follows two specific models are considered. The first model is the orthogonal model where Ψ is restricted to be the identity matrix; that is, the latent variables have no residual correlation. The second model is the oblique model, where Ψ is estimated as an unrestricted correlation matrix; that is, all residual correlations between the latent variables are estimated as free parameters. Later on the model is generalized to include structured variance–covariance matrices Ψ.

First consider the identification issues for the orthogonal model. For each orthogonal matrix H of dimension m, that is, a square matrix H such that $HH^T = I$, one can replace the η vector by $H\eta$ and obtain an equivalent model. That is because the variance $H\eta$ is again the identity matrix. Again the Λ matrix is replaced by ΛH^{-1} and similarly the rest of the parameters are changed. EFA offers a solution to this nonidentification problem. The model is identified by minimizing

$$f(\Lambda^*) = f(\Lambda H^{-1}) \tag{5}$$

over all orthogonal matrices H, where f is a function called the rotation criteria or simplicity function. Several different simplicity functions have been utilized in EFA; see Jennrich and Sampson (1966) and Appendix A. For example, the Varimax simplicity function is

$$f(\Lambda) = -\sum_{i=1}^{p}\left(\frac{1}{m}\sum_{j=1}^{m}\lambda_{ij}^4 - \left(\frac{1}{m}\sum_{j=1}^{m}\lambda_{ik}^2\right)^2\right). \tag{6}$$

These functions are usually designed so that among all equivalent Λ parameters the simplest solution is obtained.

Minimizing the simplicity function is equivalent to imposing the following constraints on the parameters Λ (see Archer & Jennrich, 1973):

$$R = ndg\left(\Lambda^T\frac{\partial f}{\partial\Lambda} - \frac{\partial f}{\partial\Lambda}^T\Lambda\right) = 0. \tag{7}$$

where the ndg refers to the nondiagonal entries of the matrix. Note that the preceding matrix is symmetric and therefore these are $m(m-1)/2$ constraints. These constraints are in addition to the $m(m+1)/2$ constraints that are directly imposed on the Ψ matrix for a total of m^2 constraints needed to identify the model.

The identification for the oblique model is developed similarly. The simplicity function

$$f(\Lambda^*) = f(\Lambda H^{-1}) \tag{8}$$

is minimized over all matrices H such that $diag(H\Psi H^T - I) = 0$; that is, matrices H such that all diagonal entries of $H\Psi H^T$ are 1. In this case minimizing the simplicity function is

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equivalent to imposing the following constraints on the parameters Λ and Ψ

$$R = ndg(\Lambda^T \frac{\partial f}{\partial \Lambda} \Psi^{-1}) = 0 \tag{9}$$

The preceding equation specifies $m(m-1)$ constraints because the matrix is not symmetric. These constraints are in addition to the m constraints that are directly imposed on the Ψ matrix for a total of m^2 constraints needed to identify the model.

Note, however, that the requirement for m^2 constraints is only a necessary condition and in some cases it might be insufficient. A simple implicit method for determining model identifiability is to compute the Fisher information matrix. In most cases the model is identified if and only if the Fisher information matrix is not singular (see Section 4.7.4 in Silvey, 1970). This method can be used in the ESEM framework as well. The identification of the rotated solution is established by computing the bordered information matrix (see Appendix C), which is algebraically equivalent to the Fisher information matrix. The rotated solution is identified if and only if the bordered information matrix is not singular. An overview of alternative explicit methods for establishing identifiability is given in Hayashi and Marcoulides (2006).

If the dependent variables are on different scales the elements in the Λ matrix will also be on different scales, which in turn can lead to imbalance of the minimization of the simplicity function and consequently lead to a suboptimal Λ^* solution. In EFA this issue is resolved by performing a standardization of the parameters before the rotation. Let Σ_d be a diagonal matrix of dimension p where the ith diagonal entry is the standard deviation of the Y_i variable. The standardized parameters Λ are then $\Sigma_d^{-1}\Lambda$; that is, in EFA:

$$f(\Sigma_d^{-1}\Lambda H^{-1}) \tag{10}$$

is minimized over all oblique or orthogonal matrices H. An equivalent way of conducting the EFA is to first standardize all dependent variables so that they have 0 mean and variance 1 and then complete the rotation analysis using the unstandardized Λ matrix. Alternative standardization techniques are described in Appendix B.

The structural equation model (1-2) is similarly standardized to avoid any undesired effects from large variation in the scales of the dependent variables. Define the diagonal matrix

$$\Sigma_d = \sqrt{diag(\Lambda\Psi\Lambda^T + \Theta)} \tag{11}$$

and the normalized loadings matrix Λ_0 as

$$\Lambda_0 = \Sigma_d^{-1}\Lambda. \tag{12}$$

The simplicity function

$$f(\Lambda_0 H^{-1}) \tag{13}$$

is then minimized over all oblique or orthogonal matrices H. Denote the optimal matrix H by H^*. Call this matrix the rotation matrix. Denote the optimal Λ_0 by Λ_0^*. Call Λ_0^* the rotated

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standardized solution. Note that after the rotation the optimal Λ^* matrix should be obtained in the original scale of the dependent variables

$$\Lambda^* = \Sigma_d \Lambda_0^*. \tag{14}$$

Note here that formally speaking the squares of the diagonal entries of Σ_d are not the variances of Y_i. That is because the standardization factor as defined in Equation 11 does not include the remaining part of the structural model such as the independent variables X as well as Equation 2. Nevertheless the simpler standardization factor defined in Equation 11 will reduce generally any discrepancies in the scales of the dependent variables. In addition, Equation 11 simplifies the computation of the asymptotic distribution of the parameters because it does not include the variance covariance of the independent variables X, which typically is not part of the model. The model usually includes conditional on X distributional assumptions and estimation only for the dependent variables. Note also that if the model does not include any covariates or other structural equations (i.e., if the model is equivalent to the standard EFA model) then the standardization factor Σ_d is the standard deviation just like in EFA.

The exploratory structural equation model described earlier can be estimated simply by constrained maximum likelihood estimation. This however, is not the algorithm implemented in M*plus*. The parameter constraints (9-7) are rather complicated and constrained maximization is more prone to convergence problems in such situations. The algorithm used in M*plus* is based on the gradient projection algorithm (GPA) developed in Jennrich (2001, 2002).

In the traditional EFA, the rotation of the factors affects only the parameters Λ and the Ψ matrix. In the exploratory structural equation model described earlier nearly all parameters are adjusted after the optimal rotation H^* is determined. The following formulas describe how the rotated parameters are obtained:

$$\nu^* = \nu \tag{15}$$

$$\Lambda^* = \Lambda(H^*)^{-1} \tag{16}$$

$$K^* = K \tag{17}$$

$$\Theta^* = \Theta \tag{18}$$

$$\alpha^* = H^*\alpha \tag{19}$$

$$B^* = H^* B(H^*)^{-1} \tag{20}$$

$$\Gamma^* = H^*\Gamma \tag{21}$$

$$\Psi^* = (H^*)^T \Psi H^* \tag{22}$$

Note also that in selecting the optimal factor rotation H^* we only use the measurement part of the model (i.e., only the Λ_0 parameter), which is computed from the Λ, Ψ, and Θ parameters. In this treatment the focus is on simplifying the loadings structure with the rotation. The structural part of the model is subsequently rotated but in this treatment the rotation does not simplify the structural part of the model in any way.

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Alternative approaches that somehow incorporate all structural parameters are possible, but such an approach would lead to additional computational complexities that might be difficult to justify. In addition, such an approach would be difficult to interpret. The rotation is designed to simplify the loading structure so that the factors have a clear interpretation. The structural parameters, on the other hand, are not a target for simplification. Typically we are interested in more significant coefficients among B, Γ, and K and are not interested in producing as few as possible significant coefficients using the rotation.

GENERAL EXPLORATORY STRUCTURAL EQUATION MODEL

The general ESEM model is described again by the equations

$$Y = \nu + \Lambda\eta + KX + \varepsilon \tag{23}$$

$$\eta = \alpha + B\eta + \Gamma X + \zeta \tag{24}$$

where the factors η_i can be divided in two groups, exploratory factors and confirmatory factors. Let $\eta_1, \eta_2, \ldots, \eta_r$ be the exploratory factors and $\eta_{r+1}, \ldots, \eta_m$ be the confirmatory factors. The confirmatory factors are identified the same way factors are identified in traditional SEM models, for example, by having different factor indicator variables for each of the factors. The group of exploratory factors is further divided into blocks of exploratory latent variables that are measured simultaneously. Suppose that a block of exploratory latent variables consists of $\eta_1, \eta_2, \ldots, \eta_b$. For each exploratory block a block of dependent factor indicator variables are assigned. Suppose the $Y_1, Y_2, \ldots, Y_c$ are the indicator variables assigned to the exploratory block. Note that different exploratory blocks can use the same factor indicators. Similarly exploratory factors can use the same factor indicators as confirmatory factors. The measurement model for $\eta_1, \eta_2, \ldots, \eta_b$ based on the indicators $Y_1, Y_2, \ldots, Y_c$ is now based and identified as the model in the previous section, using an optimal rotation for the exploratory factor block. Equation 24 uses all the confirmatory and exploratory factors. If H^* represents a combined optimal rotation matrix that consists of the optimal rotations for each of the exploratory factor blocks, the rotated estimates are obtained from the set of unidentified parameters again via Equations 15 through 22.

There are certain restrictions that are necessary to impose on the flexibility of this model. Exploratory factors have to be simultaneously appearing in a regression or correlated with. For example, if a factor in an exploratory block is regressed on a covariate X_i all other factors in that block have to be regressed on that covariate. Similarly, if a variable is correlated with an exploratory factor, the variable has to be correlated to all other variables in that exploratory block; that is, these covariance parameters can either be simultaneously 0 or they have to be simultaneously free and unconstrained.

ESTIMATION

This section describes the procedure used to estimate the ESEM model, including the estimates for the asymptotic distribution of the parameter estimates. The estimation consists of several

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steps. In the first step using the maximum likelihood estimator an SEM model is estimated where for each exploratory factor block the factor variance–covariance matrix is specified as $\Psi = I$, giving $m(m+1)/2$ restrictions, and the exploratory factor loading matrix for the block has all entries above the main diagonal, in the upper right corner, fixed to 0, giving the remaining $m(m-1)/2$ identifying restrictions. This model is referred to as the starting value model or the initial model or the unrotated model. It is well known that such a model can be subsequently rotated into any other exploratory factor model with m factors. The asymptotic distribution of all parameter estimates in this starting value model is also obtained. Then for each exploratory block or simple ESEM model the variance–covariance matrix implied for the dependent variable based only on

$$\Lambda\Lambda^T + \Theta \tag{25}$$

and ignoring the remaining part of the model is computed. The correlation matrix is also computed and using the delta method the asymptotic distribution of the correlation matrix and the standardization factors are obtained. In addition, using the delta method again the joint asymptotic distribution of the correlation matrix, standardization factors, and all remaining parameters in the model are computed. A method developed in Asparouhov and Muthén (2007) is then used to obtain the standardized rotated solution based on the correlation matrix and its asymptotic distribution. See Appendix C for a summary of this method. This method is also extended to provide the asymptotic covariance of the standardized rotated solution, standardized unrotated solution, standardization factors, and all other parameters in the model. This asymptotic covariance is then used to compute the asymptotic distribution of the optimal rotation matrix H and all unrotated model parameters. The optimal rotation matrix H is computed as follows

$$H = M_0^{-1} M_0^* \tag{26}$$

where M_0 is a square matrix that consists of the first m rows of Λ_0 and similarly M_0^* is a square matrix that consists of the first m rows of Λ_0^*. Finally all rotated parameters and their asymptotic distribution are obtained using Equations 15 through 22 and the delta method.

This estimation method is equivalent to the constrained maximum likelihood method based on Equation 7 or Equation 9. The estimation of the starting value model may give nonconvergence. A random starting value procedure is implemented in M*plus* for this estimation. In addition, a random starting value procedure is implemented in M*plus* for the rotation algorithms that are prone to multiple local minima problems.

CONSTRAINED ROTATION

Factor analysis is often concerned with invariance of measurements across different populations such as defined by gender and ethnicity (e.g., Meredith, 1993). Studies of measurement invariance and population differences in latent variable distribution are commonplace through multiple-group analysis (Jöreskog & Sörbom, 1979). A similar situation occurs for longitudinal data where measurement invariance is postulated for a factor model at each of the time points. Analysis of measurement invariance, however, has been developed and used only with CFA

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measurement specifications. Although related methods have been proposed in EFA settings (see Meredith, 1964, and Cliff, 1966), they only attempt to rotate to similar factor patterns. The methods of this article introduce multiple-group EFA and multiple-group analysis of EFA measurement parts in SEM. This development makes it possible for a researcher to not have to move from EFA to CFA when studying measurement invariance.

This section describes ESEM models constraining the loadings to be equal across two or more sets of EFA blocks. For example, in multiple-group analysis it is of interest to evaluate a model where the loading matrices are constrained to be equal across the different groups. This can easily be achieved in the ESEM framework by first estimating an unrotated solution with all loadings constrained to be equal across the groups. If the starting solutions in the rotation algorithm are the same, and no loading standardizing is used, the optimal rotation matrix will be the same, and in turn the rotated solutions will also be the same. Thus obtaining a model with invariant rotated Λ^* amounts to simply estimating a model with invariant unrotated Λ, which that is a standard task in maximum likelihood estimation.[2]

When an oblique rotation is used, an important modeling possibility is to have the Ψ matrix also be invariant across the groups or alternatively to be varying across the groups. These models are obtained as follows.[3] To obtain varying Ψ across the groups one simply estimates an unrotated solution with $\Psi = I$ in the first group and an unrestricted Ψ matrix in all other groups. Note that unrestricted here means that Ψ is not a correlation matrix but the variances of the factors are also free to be estimated. It is not possible in this framework to estimate a model where in the subsequent groups the Ψ matrix is an unrestricted correlation matrix, because even if in the unrotated solution the variances of the factors are constrained to be 1, in the rotated solution they will not be 1. However, it is possible to estimate an unrestricted variance Ψ in all but the first group and after the rotation the rotated Ψ will also be varying across groups.

Similarly, when the rotated and unrotated loadings are invariant across groups one can estimate two different models in regard to the factor intercept and the structural regression coefficients. These coefficients can also be invariant or varying across groups simply by estimating the invariant or group-varying unrotated model. Note that in this framework only full invariance can be estimated; that is, it is not possible to have measurement invariance for one EFA factor but not for the other if the two EFA factors belong to the same EFA block. Similar restrictions apply to the factor variance covariance, intercepts, and regression coefficients. If the model contains both EFA factors and CFA factors all of the usual possibilities for the CFA factors are available.

ROTATION CRITERIA

When the EFA specification is used in ESEM instead of CFA the choice of the rotation procedure becomes important. This section considers the properties of some key rotation

[2]Note again, however, that M*plus* will automatically use RowStandardization=Covariance, so that differences across groups in the residual variances Θ do not cause differences in the rotated solutions (see Appendix B).

[3]Using again RowStandardization=Covariance the estimated unrotated solution with equality of the loadings across groups and all $\Psi = I$ leads to a rotated solution with equality in the rotated loadings as well as in the Ψ matrix (see Appendix B).

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criteria: Quartimin, Geomin, and the Target criteria. Further rotation criteria are given in Appendix A.[4]

The choice of the rotation criterion is to some extent still an open research area. Generally it is not known what loading matrix structures are preserved by each rotation criterion. The simulation studies presented in this article, however, indicate that the Geomin criterion is the most promising rotation criterion when little is known about the true loading structure.[5] Geomin appears to be working very well for simple and moderately complicated loading matrix structures. However, it fails for more complicated loading matrix structures involving three or more factors and variables with complexity 3 and more; that is, variables with three or more nonzero loadings. Some examples are given in the simulation studies described later. For more complicated examples the Target rotation criterion will lead to better results. Additional discussion on the choice of the rotation criterion is presented later.

Following are some general facts about rotation criteria. Let f be a rotation criterion, Λ_0 be the loading matrix, and Ψ be the factor covariance. The oblique rotation algorithm minimizes

$$f(\Lambda) = f(\Lambda_0 H^{-1}) \tag{27}$$

over all matrices H such that $diag(H\Psi H^T) = 1$, whereas the orthogonal rotation algorithm minimizes Equation 27 over all orthogonal matrices H. The matrix Λ_0 is called an $f-$ invariant loading structure if Equation 27 is minimized at $H = I$; that is, Equation 27 is minimized at the loading matrix Λ_0 itself, regardless of the value of Ψ. The invariant structures presented here are the ones that attain the global unconstrained minimum for the rotation criteria. Typically the global unconstrained rotation function minimum is 0. If Λ_0 is the true simple structure, rotations based on f will lead to Λ_0 regardless of the starting solution. There is a second requirement for this to happen; namely, Λ_0 has to be the unique minimum of f, up to a sign change in each factor and factor permutation. If it is not, the rotation algorithm will have multiple solutions and generally speaking the rotation algorithm may not be identified sufficiently.

A sufficient condition for rotation identification has been described in Howe (1955), Jöreskog (1979) and Mulaik and Millsap (2000). Consider a factor analysis model with m factors. In general, m^2 restrictions have to be imposed on the parameters in Λ and Ψ for identification purposes. For oblique rotation m factor variances are fixed to 1 and therefore additional $m(m-1)$ constraints have to be imposed. It should be noted that not all sets of $m(m-1)$ constraints will lead to identification. Consider the case when the constraints are simply $m(m-1)$ loading parameters fixed at 0. The following two conditions are sufficient conditions for rotation identifiability.

1. Each column of Λ has $m-1$ entries specified as zeroes.
2. Each submatrix Λ_s, $s = 1, \ldots, m$, of Λ composed of the rows of Λ that have fixed zeros in the sth column must have rank $m-1$.

These conditions are sufficient for rotation identification purposes regardless of what the value of the correlation matrix Ψ is. Conditions 1 and 2 can also be used to establish identifi-

[4]All of these rotation criteria are implemented in M*plus*.

[5]The Geomin rotation is now the default rotation criterion in M*plus*.

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ability of the rotation criteria. Rotation functions are generally designed so that the optimally rotated loading matrix has many zero loadings. If these zero loadings satisfy conditions 1 and 2 then the rotation method is sufficiently identified. This approach will be used with the Geomin and the Target rotation methods.

Identifiability of the rotated solution of an ESEM model can be broken into two parts. First, the unrotated solution has to be uniquely identified. Second, the optimal rotation has to be uniquely identified. Conditions 1 and 2 can only be used to establish the identifiability of the optimal rotation, but they cannot be used to establish identifiability of the unrotated solution (see Bollen & Jöreskog, 1985). The implicit information matrix method can be used to establish identifiability for each of the two parts. If the information matrix of the unrotated solution is not singular then the unrotated solution is identified. If the bordered information matrix (see Appendix C), is also not singular, then the optimal rotation is also uniquely identified, and therefore the ESEM model is uniquely identified as well.

In general one needs to know what structures are invariant under which rotation criteria so that one can make a proper rotation criterion selection for the type of structure for which one is searching. In the next three sections the Quartimin, Geomin, and Target rotation criteria and their invariant loading structures are described. Let the loading matrix Λ be a matrix with dimensions p and m.

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Quartimin

The rotation function for the Quartimin criterion is

$$f(\Lambda) = \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l \neq j}^{m} \lambda_{ij}^2 \lambda_{il}^2. \tag{28}$$

If each variable loads on only one factor (i.e., each row in Λ has only one nonzero entry), then Λ is Quartimin invariant, and this rotation criterion will work perfectly for recovering such a factor loading structure. Note that in this case the minimum of the rotation function is the absolute minimal value of 0. Note also that this fact is independent of the number of variables or the number of factors. Usually no other rotation criteria can be as effective as Quartimin for these kinds of simple loading structures in terms of Mean Squared Error (MSE) of the parameter estimates. However, rotation criteria such as Geomin will generally produce rotation results similar to Quartimin.

Geomin

The rotation function for the Geomin rotation criterion is

$$f(\Lambda) = \sum_{i=1}^{p} \left(\prod_{j=1}^{m} (\lambda_{ij}^2 + \epsilon) \right)^{1/m} \tag{29}$$

where ϵ is a small constant. The original purpose of this constant is to make the rotation function differentiable when there are zero loadings, but by varying the constant one can actually create different rotation criteria.

Note that if $\epsilon = 0$ and one entry in each row is zero, the Geomin rotation function is zero; that is, the rotation function is already minimized and the minimization process does not help in the identification of the remaining entries in the row. If, however, $\epsilon > 0$ this problem is resolved to some extent. Note also that the Geomin rotation function is simply the sum of the rotation functions for each of the rows, but the rotation function for each row cannot be minimized separately because the loading parameters are not independent across rows. They can only vary according to an oblique or orthogonal rotation. Thus even when $\epsilon = 0$ and each row contains a zero, the nonzero entries in the row can be identified through the sufficient conditions 1 and 2 earlier.

The known Geomin-invariant loading structures are now described. Consider first the case when the parameter ϵ is 0 (or a very small number such as 10^{-5}). The Geomin function is 0 for all Λ structures that have at least one 0 in each row; that is, structures with at least one zero in each row are Geomin invariant. This is a very large set of loading structures. However, in many cases there is more than one equivalent Λ structure with at least one zero in each row. Suppose that $p \geq m(m-1)$ for oblique rotations (and $p \geq m(m-1)/2$ for orthogonal rotations) where p is the number of dependent variables and m is the number of factors and that the sufficient conditions 1 and 2 are satisfied. Then the Λ structure is unique and will therefore be completely recovered by the Geomin criterion. Even in this case however, there could be multiple solutions that reach the 0 rotation function value because a different set of 0 locations can lead to a different rotated solution.

The Geomin rotation criterion is known to frequently produce multiple solutions, or multiple local minima with similar rotation function values. The role of the ϵ value is to improve the shape of the rotation function, so that it is easier to minimize and to reduce the number of local solutions. Models with more factors are more likely to have more local solutions and are more difficult to minimize. Thus larger ϵ values are typically used for models with more factors.[6] Note, however, that multiple solutions is not a problem but rather an opportunity for the analysis. See Rozeboom (1992) and the rotation choice later in this article.

Another reason to include a positive ϵ value in the Geomin rotation function is the fact that if $\epsilon = 0$ the rotation function is not differentiable. Differentiability is important for convergence purposes as well as standard error estimation. For example, if $\epsilon < 10^{-5}$, the convergence can be very slow and the prespecified maximum number of iterations can be exceeded.

Target

Conceptually, target rotation can be said to lie in between the mechanical approach of EFA rotation and the hypothesis-driven CFA model specification. In line with CFA, target loading values are typically zeros representing substantively motivated restrictions. Although the targets influence the final rotated solution, the targets are not fixed values as in CFA, but zero targets can end up large if they do not provide good fit. An overview with further references is given in Browne (2001), including reference to early work by Tucker (1944).

[6]The M*plus* default for ϵ for two factors is 0.0001, for three factors is 0.001, and for four or more factors it is 0.01.

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The target rotation criterion is designed to find a rotated solution Λ^* that is closest to a prespecified matrix B. Not all entries in matrix B need to be specified. For identification purposes at least $m-1$ entries have to be specified in each column for oblique rotation and $(m-1)/2$ entries have to be specified in each column for orthogonal rotation. The rotation function is

$$f(\Lambda) = \sum_{i=1}^{p} \sum_{j=1}^{m} a_{ij} (\lambda_{ij} - b_{ij})^2 \tag{30}$$

where a_{ij} is either 1 if b_{ij} is specified or 0 if b_{ij} is not specified. The most common specification choice for b_{ij} is 0. Specifying many of the target loadings as 0 can be a very useful and effective way to rotate the loading structure into a hypothesized simple structure.

The known Target invariant loading structures can be described as follows. If all targets in the rotation function are correct then the Λ matrix minimizes the rotation criteria. In addition, if at least $m(m-1)$ zero targets are specified that satisfy the sufficient conditions 1 and 2[7] then the Λ matrix is the unique minimum and therefore it is Target invariant.

For example, consider a three-factor EFA model with nine measurement variables. Data are generated and estimated according to this model with the following parameter values. The factor variance–covariance Ψ is the identity matrix and the loading matrix Λ is as follows:

$$\begin{pmatrix} 1 & (0) & (0) \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ (0) & 1 & (0) \\ (0) & 1 & 0 \\ 0 & 1 & 0 \\ 0 & (0) & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

The residual variances of the dependent variables are 1. The simulation study is based on 100 samples of size 1,000. The data are analyzed using an EFA model with target rotation where the targets are the entries in the parentheses in the preceding matrix

$$\lambda_{41} = \lambda_{51} = \lambda_{12} = \lambda_{72} = \lambda_{13} = \lambda_{43} = 0 \tag{32}$$

Obviously condition 1 is satisfied. Consider now the submatrices Λ_s. Because the sth column of Λ_s by definition consists of all zeroes, that column will not contribute to the rank of Λ_s and thus the sth column can be removed for simplicity. In the preceding example the

[7] M*plus* checks these conditions. If they fail, M*plus* will automatically suggest alternative targets.

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submatrices are

$$\Lambda_1 = \begin{pmatrix} \lambda_{42} & \lambda_{43} \\ \lambda_{52} & \lambda_{53} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\Lambda_2 = \begin{pmatrix} \lambda_{11} & \lambda_{13} \\ \lambda_{71} & \lambda_{73} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{41} & \lambda_{42} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The ranks of these matrices are as follows: $rank(\Lambda_1) = 1$, $rank(\Lambda_2) = 2$, $rank(\Lambda_3) = 2$. Thus the submatrix Λ_1 does not satisfy the identifying condition 2 and it has to be modified; that is, the targets in column 1 have to be modified. This is confirmed in the simulation. From the 100 samples, 13 samples recognized the model as a nonidentified model. For the remaining samples many of the parameters have large standard error estimates and generally all parameter estimates are biased. The average absolute bias for all loading parameters is 0.511. The average standard error for the loading parameters is 1.393. Such large standard errors indicate a poorly identified model.

The reason that the nonidentification is not recognized in all samples is as follows. Whereas for the true parameter values $rank(\Lambda_1) = 1$, for individual samples the $rank(\hat{\Lambda}_1)$ may actually be 2 because of variation in the data generation and thus 87 of the 100 samples were considered identified. However, that identification is very poor because $\hat{\Lambda}_1$ is generally quite close to Λ_1; that is, it is nearly singular and has deficiency in the rank.

Now consider an alternative target specification. Replace the target $\lambda_{51} = 0$ with the target $\lambda_{71} = 0$. All other targets remain the same. The new submatrix Λ_1 now is

$$\Lambda_1 = \begin{pmatrix} \lambda_{42} & \lambda_{43} \\ \lambda_{72} & \lambda_{73} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which clearly has rank 2 and the model is now well identified. The results of the simulation confirm this. The average absolute bias for the loading estimates is now 0.003, and the average standard error for the loading estimates is 0.039.

Note that conditions 1 and 2 are generally only sufficient conditions for identification. These conditions are strictly speaking not necessary. A necessary condition is the fact that there should be at least $m(m-1)$ targets, because that will lead to the $m(m-1)$ constraints needed for identification purposes. The preceding simulation example, however, suggests that for practical purposes one could treat conditions 1 and 2 also as necessary conditions.

For orthogonal rotations the identification requirements are similar, however, now only $(m-1)/2$ targets should be specified in each column, because the Ψ matrix has $m(m-1)/2$ additional constraints, beyond the m factor variances fixed at 1. If m is even $(m-1)/2$ is not an integer, so in that case the total number of targets has to be at least $m(m-1)/2$ while each column can contain a different number of targets. Again, however, all submatrices Λ_s have to be of full rank.

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AN EMPIRICAL EXAMPLE

An example is analyzed to highlight both the EFA extensions and the SEM extensions made possible with ESEM. The example concerns a teacher-rated measurement instrument capturing aggressive and disruptive behavior among a sample of U.S. students in Baltimore public schools (Ialongo et al., 1999). A total of 248 girls and 261 boys were observed in 27 classrooms over Grades 1 through 3. The instrument consists of 13 items scored as 1 (*almost never*) through 6 (*almost always*). A first analysis considers Grade 3 gender differences in the factors behind the 13 items, using multiple-group EFA to study measurement invariance and differences in factor means, variances, and covariances. A second analysis studies antecedents of Grade 3 aggressive and disruptive behavior where the exploratory factors are related to covariates, in this case Grade 1 factors for aggressive and disruptive behavior and a poverty index. Several additional latent variable analysis features are illustrated that are new in the context of an exploratory measurement structure. First, the items are treated as continuous normal variables in the estimation, but due to the skewed distributions, nonnormality robust χ^2 model testing and standard errors will be used.[8] Second, the data are hierarchical with students nested within classrooms so that χ^2 model testing and standard errors that also take the cluster sample feature into account are used[9] (for an overview of these techniques, see Asparouhov & Muthén, 2005). An alternative modeling approach for this example that includes classroom-level modeling and not just cluster sampling adjustments is presented in Muthén and Asparouhov (2008). Third, the analysis involves using Lagrangian multipliers (modification indexes) to search for sources of model misfit in the presence of nonnormality and hierarchical data.

These two examples have model features that have not been possible to accommodate until now. In the first example, a simultaneous EFA with factor loading rotation is performed in several groups, testing different degrees of across-group invariance of measurement and factor distribution parameter arrays. In the second example, a structural equation model is formulated for a measurement model at two time points, testing EFA measurement invariance across time and also allowing the estimation of rotated structural regression coefficients. These new possibilities represent path-breaking additions to EFA and SEM.

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Multiple-Group EFA of Gender Invariance

The design of the measurement instrument suggests that three factors related to aggressive and disruptive behavior in the classroom can be expected: verbal aggression, person aggression, and property aggression. Strong gender differences are expected. Separate analyses of females and males find that a three-factor exploratory structure fit the data reasonably well. A two-group analysis of females and males with no equality restrictions across groups combines these two analyses and results in $\chi^2 = 145$ with 84 *df*, nonnormality scaling correction factor $c = 1.416$, comparative fit index (CFI) = 0.972, and root mean squared error of approximation (RMSEA) = 0.053. Adding factor loading matrix invariance results in $\chi^2 = 191$ with 114 *df*, nonnormality scaling correction factor $c = 1.604$, CFI = 0.964, and RMSEA = 0.052. A χ^2 difference test does not reject the added loading invariance hypothesis at the 1% level,

[8]This uses the M*plus* MLR estimator.

[9]This uses the M*plus* Type = Complex feature.

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TABLE 1
Two-Group Exploratory Factor Analysis Estimates for Grade 3 Aggressive and Disruptive Behavior

Items	*Verbal*	*Person*	*Property*
Stubborn	**1.19**	0.00	−0.01
Breaks rules	**0.73**	0.22	0.01
Harms others and property	0.01	**0.43**	**0.18**
Breaks things	−0.02	0.01	**0.31**
Yells at others	**0.94**	0.19	−0.03
Takes others' property	**0.36**	0.02	**0.25**
Fights	**0.36**	**0.62**	−0.02
Harms property	0.13	0.03	**0.36**
Lies	**0.77**	0.00	**0.18**
Talks back to adults	**0.87**	−0.03	**0.17**
Teases classmates	**0.58**	**0.34**	0.02
Fights with classmates	**0.42**	**0.49**	0.03
Loses temper	**0.87**	0.15	−0.00
Females			
Factor means	0.00	0.00	0.00
Factor variances	1.00	1.00	1.00
Factor correlations			
F2	0.76		
F3	0.38	0.61	
Males			
Factor means	0.35	0.69	0.80
Factor variances	1.18	2.70	5.75
Factor correlations			
F2	0.54		
F3	0.52	0.65	

Note. Values in bold are significant at the 5% level.

$\chi^2 = 47$ with 30 *df* ($p = .02$).[10] Adding measurement intercept invariance to the loading invariance gives $\chi^2 = 248$ with 124 *df*, nonnormality scaling correction factor $c = 1.517$, CFI = 0.942, and RMSEA = 0.063. A χ^2 difference test clearly rejects the added intercept invariance hypothesis, $\chi^2 = 133$ with 10 *df*. The modification indexes (MI) for the model with intercept invariance point to especially strong noninvariance for the item *breaks rules*, with MI = 18. The expected parameter change value for this parameter indicates that males have a significantly higher intercept, that is, a higher expected score given the factor value. Letting the intercept for *breaks rules* be different across gender while testing for gender invariance of the factor covariance matrix leads to a strong rejection by the χ^2 difference test, $\chi^2 = 191$ with 6 *df*. The Geomin-rotated solution for the model with invariant loadings, invariant intercepts except for *break rules*, and noninvariant factor covariance matrix is presented in Table 1. Here the ϵ value for the Geomin criterion is $\epsilon = 0.001$.

Table 1 shows that the factor loadings give a clear interpretation of the factors in terms of verbal-, person-, and property-related aggressive and disruptive behavior. Note that the

[10]The χ^2 difference testing using MLR is done as shown at www.statmodel.com/chidiff.shtml

loading estimates are not in the usual EFA metric, but correspond to items that are not standardized to unit variance and where the variances vary across items. For males the factors are also not standardized to unit variances.[11] Several items have significant cross-loadings, indicating that a simple structure CFA is not suitable. In terms of the factor distributions, males have significantly higher means on all factors and are also more heterogeneous on all factors except verbal. It is interesting to note that much of the attention in factor-analytic group comparisons is focused on factor loading similarity, with less or no attention paid to the measurement intercepts. With invariant loadings, scores consisting of sums of items with large loadings are often used as proxies for the factors. If the intercepts are not invariant, however, the use of such scores gives a distorted view of group differences. This distortion is avoided in this analysis focusing on factor mean differences under partial measurement invariance.

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Multiple-Group SEM with a Time-Invariant EFA Measurement Structure

In this section, the previous two-group, three-factor EFA model is expanded into a two-group structural equation model by regressing the Grade 3 factors on the corresponding Grade 1 factors. A covariate *lunch* is also added that predicts the three factors at both time points, where *lunch* is a dichotomous student family poverty index (free lunch recipient). Adding to the Grade 3 measurement model for females and males, measurement invariance is specified with respect to the factor loadings across the two grades. For simplicity, across-grade invariance is not specified for the measurement intercepts, and the study of factor mean differences across grade is not considered here. Factor covariance matrices are allowed to vary across the grades. This model results in $\chi^2 = 998$ with 637 *df*, nonnormality scaling correction factor $c = 1.382$, CFI $= 0.945$, and RMSEA $= 0.041$. A χ^2 difference test of across-grade loading invariance does not show a strong indication of factor loading noninvariance, resulting in $\chi^2 = 49$ with 29 *df* and $p = .01$. Geomin rotation gives a factor loading pattern similar to that of the two-group EFA for Grade 3 in Table 1.

Interesting gender differences emerge in the factor relationships across grades. For females the three Grade 1 factors do not significantly predict the three Grade 3 factors, but for males the verbal- and person-related factors have significant and positive relations over the grades. For females, the *lunch* poverty index has no significant effect on the factors at either grade, whereas for males *lunch* has a significant positive effect on the verbal and person factors in Grade 1.

In this framework it is not possible to regress only one of the exploratory factors on the poverty index variable. All three factors have to be regressed on that variable. This is necessary because even if only one factor is regressed on the poverty index variable after the rotation all three rotated factors will have nonzero regression coefficients. Similarly, each of the Grade 3 factors has to be regressed on each of the Grade 1 factors rather than only on its corresponding factor. Note also that in this example the regression coefficients of the Grade 3 factors on the Grade 1 factors are subject to rotation twice (see Equation 20), once to rotate the Grade 1 factors and a second time to rotate the Grade 3 factors.

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[11] M*plus* also provides a standardized solution. This results in different loadings across groups due to different group variances for items and factors

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SIMULATION STUDIES

A series of simulation studies is now presented to illustrate the performance of the ESEM analysis. General considerations of the use of simulation studies with EFA and ESEM are presented in Appendix D. The simulation studies are conducted with M*plus* 5.1. The M*plus* input for the first simulation is given in Appendix E.[12]

Small Cross-Loadings

One of the advantages of ESEM is that small cross-loadings do not need to be eliminated from the model. Given the lack of standard errors for the rotated solution in most EFA software, common EFA modeling practice is to ignore all loadings below a certain threshold value such as 0.3 on a standardized scale (see Cudeck & O'Dell, 1994). In subsequent CFA such loadings are typically fixed to 0 (e.g., van Prooijen & van der Kloot, 2001). Small model misspecifications such as these, however, can have a relatively large impact on the rest of the model.

In the following simulation study data are generated according to a two-factor model with 10 indicator variables Y_j and one covariate X. Denote the two factors by η_1 and η_2. The model is specified by the following two equations

$$Y = \nu + \Lambda\eta + \varepsilon \tag{33}$$

$$\eta = B\ X + \zeta \tag{34}$$

where ε is a zero mean normally distributed residual with covariance matrix Θ and ζ are zero mean normally distributed residuals with covariance matrix Ψ. The following parameter values are used to generate the data. The intercept parameter $\nu = 0$, the residual covariance Θ is a diagonal matrix with the value 0.36 on the diagonal. The loading matrix Λ is

$$\Lambda = \begin{pmatrix} 0.8 & 0 \\ 0.8 & 0 \\ 0.8 & 0 \\ 0.8 & 0.25 \\ 0.8 & 0.25 \\ 0 & 0.8 \\ 0 & 0.8 \\ 0 & 0.8 \\ 0 & 0.8 \\ 0 & 0.8 \end{pmatrix} \tag{35}$$

The values $\lambda_{42} = \lambda_{52} = 0.25$ represent the small cross-loadings. The true value for Ψ is

$$\Psi = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

[12]A tutorial on M*plus* simulation studies with ESEM is available in M*plus* 5.1, Examples Addendum available at www.statmodel.com/ugexcerpts.shtml. In addition, all M*plus* input and outputs for the simulation studies presented in this article are available by e-mail from the second author (bmuthen@ucla.edu).

The true values for the regression slopes are

$$B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}.$$

The covariate X has a standard normal distribution. The simulation study uses 100 samples of size 1,000. The samples are then analyzed by ESEM based on Geomin rotation with $\epsilon = 0.0001$, ESEM based on Geomin rotation with $\epsilon = 0.01$, ESEM based on Quartimin rotation, and the CFA-SEM model where the two cross-loadings λ_{42} and λ_{52} are held fixed to 0. All methods produced unbiased estimates for ν and Θ parameters. The results for the remaining parameters are presented in Tables 2 and 3.

It is clear from these results that the consequences of eliminating small cross-loadings in the SEM analysis can result in substantial bias in the rest of the parameter estimates as well as poor confidence interval coverage. Among the three ESEM methods the best results were obtained by the Geomin method with $\epsilon = 0.0001$. The Quartimin method and Geomin with $\epsilon = 0.01$ showed some small biases that lead to poor confidence interval coverage. In contrast, ESEM based on Geomin rotation with $\epsilon = 0.0001$ produces results with little bias for all

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TABLE 2
Comparison of ESEM and CFA-SEM with Small Cross-Loadings: Average Parameter Estimates

Parameter	*True Value*	*CFA-SEM*	*ESEM Quartimin*	*ESEM Geomin (0.01)*	*ESEM Geomin (0.0001)*
λ_{11}	0.80	0.75	0.84	0.82	0.81
λ_{21}	0.80	0.75	0.83	0.82	0.80
λ_{31}	0.80	0.75	0.83	0.82	0.81
λ_{41}	0.80	0.99	0.84	0.82	0.81
λ_{51}	0.80	0.99	0.84	0.83	0.81
λ_{61}	0.00	0.00	0.01	0.01	0.00
λ_{71}	0.00	0.00	0.01	0.01	0.00
λ_{81}	0.00	0.00	0.01	0.01	0.00
λ_{91}	0.00	0.00	0.01	0.01	0.00
λ_{101}	0.00	0.00	0.01	0.01	0.00
λ_{12}	0.00	0.00	−0.06	−0.03	−0.01
λ_{22}	0.00	0.00	−0.06	−0.03	−0.01
λ_{32}	0.00	0.00	−0.06	−0.04	−0.01
λ_{42}	0.25	0.00	0.18	0.21	0.24
λ_{52}	0.25	0.00	0.18	0.21	0.23
λ_{62}	0.80	0.80	0.80	0.80	0.80
λ_{72}	0.80	0.80	0.80	0.79	0.80
λ_{82}	0.80	0.80	0.80	0.80	0.80
λ_{92}	0.80	0.80	0.80	0.79	0.80
λ_{102}	0.80	0.80	0.80	0.80	0.80
β_1	0.50	0.61	0.56	0.54	0.52
β_2	1.00	1.00	1.00	1.00	1.00
ψ_{12}	0.50	0.61	0.55	0.53	0.51

Note. CFA-SEM = confirmatory factor analysis-structural equation modeling; ESEM = exploratory structural equation modeling.

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TABLE 3
Comparison of ESEM and CFA-SEM with Small Cross-Loadings: Confidence Intervals Coverage

Parameter	*CFA-SEM*	*ESEM Quartimin*	*ESEM Geomin (0.01)*	*ESEM Geomin (0.0001)*
λ_{11}	0.54	0.77	0.85	0.90
λ_{21}	0.48	0.87	0.97	0.97
λ_{31}	0.48	0.82	0.93	0.95
λ_{41}	0.00	0.78	0.86	0.95
λ_{51}	0.00	0.76	0.88	0.95
λ_{61}	1.00	0.98	0.97	1.00
λ_{71}	1.00	0.95	0.94	0.97
λ_{81}	1.00	0.96	0.98	1.00
λ_{91}	1.00	0.95	0.95	1.00
λ_{101}	1.00	0.95	0.92	0.97
λ_{12}	1.00	0.05	0.50	0.95
λ_{22}	1.00	0.05	0.46	0.96
λ_{32}	1.00	0.02	0.38	0.97
λ_{42}	0.00	0.24	0.66	0.91
λ_{52}	0.00	0.09	0.67	0.89
λ_{62}	0.99	0.98	0.97	0.98
λ_{72}	0.99	0.95	0.94	0.97
λ_{82}	0.94	0.96	0.96	0.96
λ_{92}	0.95	0.97	0.97	0.99
λ_{102}	0.94	0.97	0.97	0.97
β_1	0.13	0.59	0.83	0.94
β_2	0.96	0.97	0.97	0.97
ψ_{12}	0.01	0.44	0.77	0.93

Note. CFA-SEM = confirmatory factor analysis-structural equation modeling; ESEM = exploratory structural equation modeling.

parameters and coverage near the nominal 95% level. A simulation study based on samples with only 100 observations reveals very similar results to the ones presented in Tables 2 and 3; that is, these results appear to be independent of the sample size.

The chi-square test of fit for the model is also affected by the elimination of small cross-loadings. Using a simulation with 500 samples of size 1,000 the SEM model is rejected 100% of the time and the ESEM model is rejected only 7% of the time. For a sample size of 100 the rejection rate for the SEM model is 50% and for the ESEM model it is 10%. These results show that small, inconsequential cross-loadings can lead to a correct chi-square rejection of an otherwise well-constructed SEM model. Using approximate fit measures for the SEM model, such as CFI/Tucker–Lewis Index (TLI), RMSEA, and standardized root mean squared residual (SRMR), one can avoid this rejection problem to a substantial degree. Using samples of size 1,000 and the RMSEA measure with a cutoff value of 0.06 the model is rejected only 50% of the time. Using the SRMR measure with cutoff value of 0.08, the model is never rejected.

The simulation study presented here is not as easy to interpret as traditional simulation studies especially when it comes to comparing different rotation methods. To provide proper interpretation of the results one has to first accept the notion that the loading matrix presented

TABLE 4
Rotation of Population Loading Matrix

Λ		Λ_q		$\Lambda_{0.01}$		$\Lambda_{0.0001}$	
0.80	0.00	0.80	−0.07	0.82	−0.03	0.80	−0.01
0.80	0.00	0.80	−0.07	0.82	−0.03	0.80	−0.01
0.80	0.00	0.80	−0.07	0.82	−0.03	0.80	−0.01
0.80	0.25	0.80	0.18	0.82	0.21	0.80	0.24
0.80	0.25	0.80	0.18	0.82	0.21	0.80	0.24
0.00	0.80	0.01	0.83	0.01	0.79	0.00	0.80
0.00	0.80	0.01	0.83	0.01	0.79	0.00	0.80
0.00	0.80	0.01	0.83	0.01	0.79	0.00	0.80
0.00	0.80	0.01	0.84	0.01	0.79	0.00	0.80
0.00	0.80	0.01	0.84	0.01	0.79	0.00	0.80

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in Equation 35 is the simplest possible loading matrix among all rotated versions of that matrix. In particular, one has to accept the notion that Λ given in Equation 35 is simpler than rotations of Λ that have no zero loading values. If this simplicity notion is accepted, then the simulation study can be interpreted in the traditional sense; that is, the matrix Λ given in Equation 35 is the true loading matrix that has to be estimated by the rotated loading matrix $\hat{\Lambda}$. Now suppose that, for some reason, an analyst decides that another rotated version of Λ is simpler than the one given in Equation 35. In that case, the preceding simulation study would be irrelevant and a different rotation criterion, that targets the alternative rotated version of Λ, would have to be explored.

To illustrate the preceding point, consider the rotation results on the population level. Using the rotation algorithms with the true population parameters Λ, Ψ, and Θ one can obtain the optimal rotations on the population level.[13] Denote the Λ rotations obtained by Quartimin, Geomin with $\epsilon = 0.01$, and Geomin with $\epsilon = 0.0001$ by Λ_q, $\Lambda_{0.01}$, and $\Lambda_{0.0001}$, respectively. Denote the corresponding Ψ rotations by Ψ_q, $\Psi_{0.01}$, and $\Psi_{0.0001}$, respectively.[14] These matrices are presented in Tables 4 and 5. Finite sample based rotated parameter estimates are essentially consistent estimates of the rotated population values presented in Table 4 and 5. All four of these rotated solutions are equivalent in terms of model fit because the matrices are rotations of each other. To decide which rotation is optimal one has to consider the notion of simplicity. Which of the four Λ matrices should be considered the simplest and the most interpretable? Regardless of the arguments and notion of simplicity in this example, one inevitably reaches the conclusion that the matrix Λ is the simplest. Therefore in the estimation process this matrix should be considered the desired matrix. It is clear that $\Lambda_{0.0001}$ is the closest to Λ and that is the reason why the Geomin rotation with $\epsilon = 0.0001$ produced the best results in the simulation study. If, however, for some reason one decides that Λ_q is the simplest possible matrix, then obviously the Quartimin rotation would be the optimal rotation method to use. A realistic

[13]Note that Θ also influences the rotation through the correlation standardization.

[14]In M*plus* the population-level rotations are obtained by generating a large sample, such as a sample with 1,000,000 observations. In such a large sample the estimated parameters are nearly identical with the population parameters.

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TABLE 5
Rotation of Population Correlation Matrix

Ψ		Ψ_q		$\Psi_{0.01}$		$\Psi_{0.0001}$	
1.00	0.50	1.00	0.55	1.00	0.52	1.00	0.51
0.50	1.00	0.55	1.00	0.52	1.00	0.51	1.00

example where two different loading matrices are quite likely to be considered as the simplest and most interpretable is described later in this article.

Chi-Square Test of Fit and Likelihood Ratio Testing

Testing various aspects of ESEM can be done the same way as for regular SEM models. The standard chi-square test of fit that compares a structural model against an unrestricted mean and variance model can be done for ESEM the same way, using the likelihood ratio test (LRT) for the two models. For example, consider the question of how many factors are needed in the ESEM model. One standard approach is to sequentially fit models with 1, 2, ..., and so on factors and then use the smallest number of factors for which the test of fit does not reject the model. Consider the simulation example described in the previous section. Estimating the model with one factor leads to an average chi-square test of fit of 1,908 with 44 *df* and 100% rejection rate; that is, the LRT correctly identifies the one-factor ESEM model as insufficient. In contrast, for the two-factor ESEM model the average chi-square test of fit is 35 with 34 *df* and the rejection rate dropped to 9%; that is, the LRT correctly finds the two-factor ESEM model well fitting. It is possible to estimate even a three-factor ESEM model, although convergence problems occur in 30 out of the 100 replications. The average chi-square test of fit for the three-factor ESEM model is 20 and with 25 *df* this leads to a 0% rejection rate. The underestimation of the chi-square test statistic and Type I error in this case is due to overfactoring (see Hayashi, Bentler, & Yuan, 2007).

Alternatively, the LRT can be used to test directly an $m-1$-factor ESEM model against an m-factor ESEM model, without testing the models against the unrestricted mean and variance models. In the preceding example, testing the one-factor model against the two-factor model gives an average chi-square test statistic of 1,873 and with 8 *df* this leads to a 100% rejection rate. In certain cases such direct testing can be preferable as it directly tests the hypothesis of interest, namely, whether or not the additional factor is needed. The direct test will also be more powerful than the general test of fit model; that is, it will outperform the test of fit approach in small sample size problems. Note, however, that testing $m-1$ factors against m factors is susceptible to overfactoring and inflated Type I error (see Hayashi et al., 2007).

In practice, however, not all of the residual correlation will be picked up by the unrestricted loading structure of the ESEM model and strictly using the chi-square test of fit will often lead to an unreasonable number of factors in the model, many of which contribute little to the overall model fit. In such cases one can use approximate fit indexes such as SRMR, CFI, and TLI to evaluate the fit of the model. One can also use the SRMR index to evaluate the improvement in the fit due to each additional factor. For example, if an additional factor contributes less than 0.001 decrease in the SRMR, it seems unreasonable to include such factors in the model.

Instead one can use the new ESEM feature, extending the standard EFA, by including residual covariance parameters in the model in addition to the exploratory factors. Furthermore it is possible to point out which residual covariances should be included in the model, and thereby improve factor stability and overall fit, by using standard modeling tools such as modification indexes, and standardized and normalized residuals.

The LRT can be used also to test an EFA model against a CFA model. Consider again the simulation example given earlier and the LRT of that model against the CFA model based on all nonzero loadings, i.e., including the two small cross-loadings. Note first that the two models are nested. This is not very easy to see because of the parameter constraints imposed on the ESEM parameters by the rotation algorithm. There are eight loading parameters that are fixed at 0 in the CFA-SEM but not in the ESEM. However, the ESEM model has two parameter constraints, imposed by the rotation algorithm, that involve all loading and factor covariance parameters. To see that the CFA model is nested within the ESEM model, first note that the ESEM model is equivalent to its starting unrotated solution. The rotated solution has the same log-likelihood value as the unrotated starting value solution, and any testing of a model against an ESEM model is essentially a test against the unrotated starting value model. A number of different unrotated solutions can be used at this point. Two of these are generally convenient in assessing the model nesting. The first one is the orthogonal starting value where the factor variance–covariance matrix is the identity matrix and the loadings above the main diagonal in the upper right corner are all fixed to 0. The second unrotated starting value solution that can be used is the oblique starting value where the factor variances are fixed to 1, the factor covariances are free, and each loading column contains exactly $m - 1$ zeroes in locations that satisfy the sufficient condition 2. For example, a square submatrix of size m can be selected from the loading matrix and in this submatrix all values except the main diagonal entries can be fixed to 0. In the preceding example one can use the oblique starting value solution to assert the nesting of the CFA and ESEM models. The ESEM model is equivalent to an unrotated oblique starting value solution with any two loadings from different rows fixed to 0. It is now clear that the CFA model can be thought of as more constrained than the ESEM model where the additional constraints simply fix the remaining six loadings at 0.

Conducting the LRT between the ESEM and CFA models for the earlier simulation example, using 100 samples of size 5,000, the average test statistic is 5.73 and with 6 *df* that leads to a rejection rate of only 2%; that is, the LRT correctly concludes that the CFA model with all eight loadings fixed to 0 is well fitting.

Now consider the situation when both nested models are approximately fitting models; that is, the models have small misspecifications but the sample size is large enough that even small misspecifications lead to poor tests of fit. For example, if the data generation given earlier is altered by adding a residual covariance between Y_7 and Y_8 of 0.05, using a sample size of 5,000, both the ESEM and CFA models are rejected by the test of fit 100% of the time with average chi-square test of fit statistics of 88 and 97, respectively. The average SRMR measures are 0.004 and 0.005, respectively; that is, both models are fitting approximately in all 100 replications. Conducting the LRT between the CFA and ESEM models provides relatively good results here as well. The average LRT statistic for testing the CFA model against the ESEM model is 8.65 and with 6 *df*, this leads to a 14% rejection rate. This suggests that even when the models are fitting the data only approximately, the LRT can be used to distinguish between ESEM and CFA models. The relatively small inflation in the rejection rate is due

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to the fact that the more flexible ESEM model is able to accommodate more of the model misspecifications than the CFA model. The inflation, however, is relatively small and the LRT can clearly be recommended. Even though both the ESEM and CFA models are incorrect in this simulation, the LRT correctly concludes that the eight loadings are indeed 0.

Multiple-Group ESEM

This section describes a multiple-group example and demonstrates the constrained rotation technique described earlier for group-invariant loading matrices. Consider a two-group, two-factor model with 10 dependent variables:

$$Y = \nu_g + \Lambda_g \eta + \varepsilon \tag{36}$$

$$\eta = \alpha_g + \zeta \tag{37}$$

where ε and ζ are zero mean residuals with covariance matrices Θ_g and Ψ_g. One common application of multiple group analysis is to test measurement invariance across the groups; that is, to test the hypothesis $\Lambda_1 = \Lambda_2$ (see Jöreskog & Sörbom, 1979). Estimating the measurement invariance model is of interest as well. This simulation study evaluates the performance of the ESEM technique for the measurement invariance model. Data are generated using the following parameter values $\nu_1 = \nu_2 = 1$, $\alpha_1 = 0$, $\alpha_2 = (0.5, 0.8)$, Θ_1 is a diagonal matrix with all diagonal values 1, Θ_2 is a diagonal matrix with all diagonal values 2,

$$\Lambda_1 = \Lambda_2 = \begin{pmatrix} 0.8 & 0 \\ 0.8 & 0 \\ 0.8 & 0 \\ 0.8 & 0 \\ 0.8 & 0 \\ 0 & 0.8 \\ 0 & 0.8 \\ 0 & 0.8 \\ 0 & 0.8 \\ 0 & 0.8 \end{pmatrix}$$

$$\Psi_1 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

$$\Psi_2 = \begin{pmatrix} 1.5 & 1 \\ 1 & 2 \end{pmatrix}.$$

The simulation study is conducted for samples with 100 observations in each group as well as samples with 500 observations in each group. The simulation study is based on 100 samples for each of the two sample size specifications. For each of the samples the ESEM model is estimated with the following constraints. The loadings and the intercepts are held equal across

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the two groups.

$$\Lambda_1 = \Lambda_2 \tag{38}$$

$$\nu_1 = \nu_2 \tag{39}$$

In the first group the factor variances are fixed to 1 and the factor means are fixed to 0:

$$\psi_{111} = \psi_{221} = 1 \tag{40}$$

$$\alpha_1 = 0. \tag{41}$$

In addition, Θ_1 and Θ_2 are estimated as diagonal matrices, α_2 is estimated as a free vector, Ψ_2 is estimated as an unrestricted variance matrix, and Ψ_1 is estimated as an unrestricted correlation matrix. This model specification is a typical measurement invariance model. Other sets of identifying restrictions can be similarly specified. The model described earlier has a total of 54 independent parameters, 10 ν parameters, 10 Θ_1 parameters, 10 Θ_2 parameters, 20 Λ parameters, 3 Ψ_2 parameters, 2 α_2 parameters, and ψ_{121}, minus the two parameter restrictions imposed on Ψ and Λ by the rotation algorithm. The ESEM model is estimated with the Geomin rotation and $\epsilon = 0.0001$. The average estimate for some of the parameters in the model and their confidence interval coverage are reported in Table 6. For sample size 500, all parameter estimates have negligible bias and the coverage is near the nominal 95% level. For sample size 100, the coverage is near the nominal 95% level; however, some of the parameter estimates show substantial bias, namely, the factor covariance parameter in both groups.

The results in Table 6 indicate that the small sample size properties of the ESEM models may be somewhat inferior to those of traditional SEM. To investigate the small sample size parameter biases in the preceding simulation study the samples with 100 observations in each group are analyzed by the following three methods: the ESEM method with Geomin rotation and $\epsilon = 0.0001$, the ESEM method with Target rotation using all 0 loadings as targets, and the SEM with all 0 loadings fixed to 0. In practice both the ESEM-Target method and the

TABLE 6
Two-Group Exploratory Structural Equation Modeling Geomin Analysis

Parameter	*True Value*	n = *100 Average Estimate*	n = *500 Average Estimate*	n = *100 Coverage*	n = *500 Coverage*
λ_{11}	0.80	0.76	0.79	0.92	0.95
λ_{12}	0.00	0.04	0.01	0.97	0.99
ψ_{121}	0.50	0.42	0.49	0.98	0.98
ν_{11}	1.00	0.99	0.99	0.94	0.98
θ_{111}	1.00	0.97	1.00	0.93	0.99
α_{12}	0.50	0.47	0.51	0.93	0.91
α_{22}	0.80	0.81	0.82	0.96	0.95
ψ_{122}	1.00	0.92	0.98	0.92	0.96
ψ_{112}	1.50	1.58	1.50	0.92	0.96
ψ_{222}	2.00	2.03	2.02	0.93	0.95
θ_{112}	2.00	1.96	1.99	0.96	0.96

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SEM method can be used as a follow-up model to the ESEM-Geomin method. Based on the ESEM-Geomin method, the ESEM-Target model is constructed by setting all loadings that are not significantly different from 0 as targets. Similarly, the SEM model is constructed by setting all loadings that are not significantly different from 0 as loadings that are fixed to 0. Note that although the parameter estimates for ESEM-Geomin show some small sample size bias for some parameters, the standard errors produced correct coverage for all parameters; that is, when evaluating the significance of small loadings for purposes of constructing the ESEM-Target model and the SEM model, the ESEM-Geomin model will correctly point out all zero loadings.

The results of this simulation study are presented in Table 7, which contains the average parameter estimates and the mean squared error (MSE) for the parameter estimates. Small sample size results should be interpreted very cautiously. Usually there is no theoretical justification for preferring one method over another for small sample size and usually simulation studies are used to draw general conclusions. However, there is no guarantee that the results in one simulation study would be similar to the results of the same simulation study with different parameters and even in the same simulation study the results can be inconsistent. For example, in this simulation the covariance in the first group is best estimated by the SEM model, whereas the covariance in the second group is best estimated by the ESEM-Target model. Nevertheless, Table 7 seems to give general guidance for reducing small sample size biases. It appears that the additional information that ESEM-Target and SEM facilitate, namely that some loadings are small or even 0, does result in a reduction of the small sample size biases and the MSE of the parameter estimates. In addition, the SEM model does appear to have slightly smaller biases overall than the ESEM-Target method although this does not appear to be a consistent trend and for some parameters ESEM-Target produces better results. For many of the parameters the three methods produce nearly identical results. The SEM model has fewer parameters overall and thus can be expected in general to produce somewhat smaller biases and smaller MSE.

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TABLE 7
Two-Group ESEM Analysis, Small Sample Size Comparison of ESEM-Geomin, ESEM-Target, and SEM

Parameter	*True Value*	*ESEM Geomin Average Estimate*	*ESEM Target Average Estimate*	*SEM Average Estimate*	*ESEM Geomin MSE*	*ESEM Target MSE*	*SEM MSE*
λ_{11}	0.80	0.76	0.77	0.78	0.021	0.022	0.010
λ_{12}	0.00	0.04	0.02	0.00	0.014	0.012	0.000
ψ_{121}	0.50	0.42	0.45	0.48	0.021	0.014	0.012
ν_{11}	1.00	0.99	0.99	0.99	0.016	0.016	0.017
θ_{111}	1.00	0.97	0.97	0.98	0.027	0.027	0.025
α_{12}	0.50	0.47	0.48	0.49	0.044	0.043	0.041
α_{22}	0.80	0.81	0.82	0.82	0.041	0.040	0.040
ψ_{122}	1.00	0.92	0.99	1.04	0.107	0.095	0.101
ψ_{112}	1.50	1.58	1.59	1.60	0.284	0.291	0.275
ψ_{222}	2.00	2.03	2.04	2.05	0.305	0.298	0.292
θ_{112}	2.00	1.96	1.96	1.96	0.097	0.097	0.096

Note. ESEM = exploratory structural equation modeling; SEM = structural equation modeling; MSE = mean squared error.

Conducting this simulation for a sample size of 500 does not lead to any substantial difference among the three methods. Thus the differences presented in Table 7 are likely to occur only in small samples.

In addition, the usual chi-square test of fit that compares the estimated ESEM model against the unrestricted mean and variance two-group model can be used to evaluate the fit of the model. In this simulation study the model has 76 *df*. For a sample size of 100, the average test of fit statistic is 78.25 with a rejection rate at 5%. For a sample size of 500, the average test of fit statistic is 76.05 with a rejection rate at 5%. This shows that the chi-square test of fit works well for the ESEM models.

General Factor

In certain EFA applications there is one main factor on which all items load. In addition, there can be other factors that are specific to the different items. This structure is also referred to as a bifactor solution in the classic factor analysis text of Harman (1976). For example, consider a three-factor model with 10 items with the following loading matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0 & 0.5 \\ 1 & 0 & 0.5 \\ 1 & 0 & 0.5 \end{pmatrix}. \tag{42}$$

If one considers oblique rotations, there is a rotation of the preceding matrix that will have just one nonzero entry in each row.

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1.12 & 0 \\ 0 & 1.12 & 0 \\ 0 & 1.12 & 0 \\ 0 & 1.12 & 0 \\ 0 & 0 & 1.12 \\ 0 & 0 & 1.12 \\ 0 & 0 & 1.12 \end{pmatrix} \tag{43}$$

$$\Psi = \begin{pmatrix} 1 & 0.89 & 0.89 \\ 0.89 & 1 & 0.80 \\ 0.89 & 0.80 & 1 \end{pmatrix} \tag{44}$$

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Thus rotation criteria such as Quartimin that converge to complexity 1 solutions will not be able to recover the general factor structure (Equation 42). Geomin with $\epsilon = 0$ has two different optimal solutions, namely Equations 42 and 43, both leading to a rotation function value of 0. For very small positive values of ϵ one can expect this to remain so. However, as ϵ increases, the rotation function can change sufficiently so that some of these multiple solutions are no longer local solutions. As ϵ increases the rotation function value for Equation 43 will be lower because it has two zeroes in each row; that is, the loadings matrix (Equation 43) will be the global minimum and Equation 42 will be at best a local solution. In fact it is not clear whether Equation 42 will represent a local solution at all. Even with $\epsilon = 10^{-4}$ using 30 random starting values, the GPA algorithm converged to Equation 43 in all 30 replications. In general it is not easy to force a minimization algorithm to find local solutions, because minimization algorithms are designed to find global solutions. The rotation function value for Equation 43 is 0.027 and for Equation 42 it is 0.214; that is, the two solutions are of a different magnitude. If ϵ is chosen to be a smaller value, such as 10^{-6}, the rotation function values are closer, but the convergence process is substantially more difficult. Many more replications are needed for convergence and the convergence criteria have to be relaxed as well. Using $\epsilon = 10^{-6}$ again most replications converge to solution Equation 43, but another local solution is found that is different from both Equations 42 and 43. In addition, in a simulation study, even if the GPA algorithm is able to find consistently a particular local solution in all samples it is difficult to implement constraints that will always recognize that particular local solution so that when the results of the simulation are accumulated the same local solution is used. This investigation shows that relying on local Geomin solutions might not work well and that from a practical perspective the loading matrix (Equation 42) should not be considered Geomin invariant.

For orthogonal rotations, however, the loading matrix (Equation 42) is Geomin invariant. This is demonstrated in the following simulation study that compares Geomin with $\epsilon = 0.001$ with another popular rotation method, Varimax. The simulation study is based on 100 samples of size 5,000. The data are generated according to the preceding model and using the loading matrix (Equation 42). The intercept parameters $\nu = 0$, the residual variance for the indicator variables is 1, and the factor covariance matrix Ψ is the identity matrix. The results of the simulation study are presented in Table 8 for a representative set of parameters. The Geomin

TABLE 8
General Factor Exploratory Structural Equation Modeling Analysis With Orthogonal Rotation

Parameter	*True Value*	*Geomin Average*	*Varimax Average*	*Geomin Coverage*	*Varimax Coverage*
λ_{11}	1.00	1.00	0.58	0.91	0.00
λ_{12}	0.00	−0.01	0.57	1.00	0.00
λ_{13}	0.00	0.01	0.58	0.98	0.00
λ_{41}	1.00	1.01	0.90	0.96	0.71
λ_{42}	0.50	0.49	0.37	0.94	0.00
λ_{43}	0.00	0.00	0.47	0.98	0.00
λ_{81}	1.00	1.00	0.45	0.96	0.00
λ_{82}	0.00	−0.01	0.31	0.98	0.00
λ_{83}	0.50	0.50	0.96	0.97	0.00

method produces unbiased parameter estimates with good confidence interval coverage. In contrast, the Varimax method produces biased parameter estimates and poor confidence interval coverage.

The Geomin method, however, has two solutions. The first solution is given in Equation 42 and has rotation function values 0.28. The second solution

$$\Lambda = \begin{pmatrix} 0.94 & 0.33 & 0 \\ 0.94 & 0.33 & 0 \\ 0.94 & 0.33 & 0 \\ 1.06 & 0 & 0.35 \\ 1.06 & 0 & 0.35 \\ 1.06 & 0 & 0.35 \\ 1.06 & 0 & 0.35 \\ 1.06 & 0 & -0.35 \\ 1.06 & 0 & -0.35 \\ 1.06 & 0 & -0.35 \end{pmatrix} \quad (45)$$

has rotation function value 0.30. Using random starting values and the population parameters, the GPA algorithm converged to the global minimum of 0.28 about half of the time and the other half it converged to the local minimum of 0.30. When the sample size is sufficiently large, such as the 5,000 used in this simulation, there will be two solutions, but they will consistently appear in the same order; that is, the global minimum in all finite sample size replications will correspond to the global minimum solution in the population model. Thus an algorithm that always selects the global minimum will essentially always select the same solution. If however, the sample size is smaller, the global and the local solutions will switch orders across the replications, and thus an algorithm that always selects the global minimum will essentially average the two different solutions and thus render useless results.[15] A more advanced algorithm that includes a method for picking the same local solution would avoid that problem. This issue is important only in simulation studies. In single-replication studies such as real data analysis, one has to simply evaluate all local solutions and choose the one that is simplest and easiest to interpret.

When a general factor model is anticipated and oblique rotation is used, the Target rotation method might be a better alternative. The next section illustrates the Target rotation with a complex loading structure.

Complexity 3

In this section the advantages of the Target rotation are demonstrated with a complexity 3 example, that is, an example with three nonzero loadings in a row. The three methods compared in this section are the Target rotation, the Geomin rotation with $\epsilon = 0.01$, and the Geomin rotation with $\epsilon = 0.0001$. Consider a four-factor 12-indicator factor analysis model with the intercept parameter $\nu = 0$, the covariance matrices Ψ and Θ as the identity matrices, and Λ

[15] Future version of M*plus* will include tools for resolving this problem.

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as follows:

$$\Lambda = \begin{pmatrix} 1 & (0) & (0) & (0) \\ 1 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 \\ (0) & 1 & (0) & (0) \\ 0 & 1 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ (0) & (0) & 1 & (0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0.5 & 1 \\ (0) & (0) & (0) & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (46)$$

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The complexity of Y_{10} is 3. The entries in the parentheses represent the targets for the Target rotation. One easy way to select targets and avoid any identification problems is to identify pure factor indicators; that is, identify one variable for each factor that loads only on that variable just like in this example. The rank condition is then automatically satisfied. When each factor has a pure indicator one can set all zero loadings for the pure indicators as targets and the loading matrix is then Target invariant; that is, the estimates are asymptotically unbiased under the Target rotation. Tables 9 and 10 contain the results of the simulation study based on the preceding model and conducted over 100 samples of size 5,000. A representative set of loading parameters is presented in Tables 9 and 10. Both Geomin-based estimations produced biased estimates. The bias of the estimates based on Geomin with $\epsilon = 0.01$ is smaller. The coverage of

TABLE 9
Complexity 3 Exploratory Structural Equation Modeling Analyses: Average Estimates

Parameter	*True Value*	*Geomin $\epsilon = 0.0001$*	*Geomin $\epsilon = 0.01$*	*Target*
λ_{11}	1.00	1.00	1.00	1.00
λ_{12}	0.00	0.00	−0.03	0.00
λ_{13}	0.00	0.00	0.01	0.00
λ_{14}	0.00	0.00	0.00	0.00
λ_{51}	0.00	0.00	0.00	0.00
λ_{52}	1.00	1.00	0.99	1.00
λ_{53}	0.50	0.49	0.45	0.50
λ_{54}	0.00	0.00	0.00	0.00
λ_{101}	0.00	0.00	0.00	0.00
λ_{102}	0.50	0.25	0.44	0.50
λ_{103}	0.50	0.25	0.41	0.50
λ_{104}	1.00	1.12	1.01	1.01
ψ_{12}	0.00	0.00	0.03	−0.01
ψ_{34}	0.00	0.22	0.06	0.00

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TABLE 10
Complexity 3 Exploratory Structural Equation Modeling Analyses: Coverage

Parameter	*Geomin* $\epsilon = 0.0001$	*Geomin* $\epsilon = 0.01$	*Target*
λ_{11}	0.94	0.94	0.94
λ_{12}	1.00	0.12	1.00
λ_{13}	0.97	0.87	1.00
λ_{14}	0.97	0.92	1.00
λ_{51}	0.98	0.97	0.94
λ_{52}	0.98	0.97	0.99
λ_{53}	0.99	0.26	0.99
λ_{54}	0.90	0.90	0.95
λ_{101}	0.99	0.93	0.97
λ_{102}	0.50	0.22	0.95
λ_{103}	0.45	0.05	0.97
λ_{104}	0.00	0.94	0.94
ψ_{12}	0.97	0.61	0.94
ψ_{34}	0.45	0.08	0.94

the Geomin-based estimation is also quite poor. In contrast, the Target rotation shows negligible bias and coverage near the 95% nominal level.

One can investigate the source of the Geomin bias by conducting the rotation on the population values and investigating all local solutions. Using $\epsilon = 0.0001$ Geomin has more than five local solutions that have similar rotation function values. One of these solutions corresponds to Equation 46. Thus the simulation study presented here somewhat unfairly evaluates Geomin. If the algorithm included evaluation of the different local Geomin solutions and included a constraint to make the additional selection among these solutions so that the solution corresponding to Equation 46 is always selected, there would be no bias. The bias in the simulation study is caused by the fact that the average estimates really represent the average estimates among a mixed sets of local Geomin solutions, instead of the same solution. In real data examples this is essentially a nonexistent problem because one simply has to consider the various Geomin local solutions.

CHOOSING THE RIGHT ROTATION CRITERION

In most ESEM applications the choice of the rotation criterion will have little or no effect on the rotated parameter estimates. In some applications, however, the choice of the rotation criterion will be critical and in such situations one has to make a choice. This section describes the underlying principles that one can follow to make that choice.

Choosing the right rotation is essentially a postestimation decision and there is no right or wrong rotation. The goal of the rotation algorithms is to select the simplest and most interpretable loading structure. It is ultimately the analyst's choice and perception of what the simplest and most interpretable loading structure is. It is the analyst's choice of what the

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rotation criterion should be and which of the multiple rotated solutions represents the best loading structure for that particular application. Understanding the properties of the different rotation criteria will help the analyst in exploring the various rotation criteria. In particular, understanding the type of loading structures that each of the rotation criteria can reproduce (i.e., the invariant loading structures) is essential.

Estimation methods based on fit function optimizations such as the maximum likelihood and least squares estimation methods would only accept the global optimum as the proper solution. Local optima are perceived as estimation problems that have to be resolved so that the global optimum is always obtained. This is not the case, however, when it comes to local minima for the rotation criteria. Understanding and exploring the ability of rotation criteria such as Geomin to produce multiple optimal solutions can help the analyst in finding the best loading structure. It will generally be useful to consider the alternative top two or three Geomin solutions when such solutions are available.[16] Similarly, changing the ϵ value in Geomin is equivalent to changing the rotation criterion. There is no correct or incorrect ϵ value. Different values for this parameter produce different rotation criteria that can enable the analyst to fine-tune the loading matrix. In fact it is important that the analyst explores the sensitivity of the Geomin solution with respect to the ϵ value. In particular ϵ values such as $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$ should always be used.

To summarize, there is no statistical reason to prefer one rotation criterion over another, one ϵ value over another, or one local minimum over another. It is entirely in the hands of the analyst to make the choice and interpret the results. It is not the data that decide what a simple loading structure is, it is not the estimator, and it is not the rotation method. The analyst alone has to decide that. Although for many simple loading structures, such as Equation 31, most analysts will agree that no alternative rotation of Λ is simpler and more interpretable, that is not the case for other loading structures such as Equation 42 and 43. For more complicated loading structures analysts can disagree on what the simplest loading structure is, even when the same rotation criterion is used and different local minima are selected. There is no statistical tool to resolve such disagreement and multiple equally valid solutions can be used.

DISCUSSION

This article has presented a new approach to SEM that extends the types of measurement models that can be used. Adding the possibility of an EFA measurement specification, strict loading restrictions in line with CFA are not necessary. The resulting ESEM approach has the full generality of regular SEM. From an EFA perspective, this implies that EFA can be performed while allowing correlated residuals, covariates including direct effects on the factor indicators, longitudinal EFA with across-time invariance testing, and multiple-group EFA with across-group invariance testing. Several factor loading rotation methods are available, including Geomin and Target rotation.

[16] M*plus* will automatically run 30 random starting values with the Geomin rotation. More random starting values can be requested using the *rstarts*= command. In addition the different rotation values are presented in regular EFA, as well as the loading structures for the different local minima. The ESEM output in M*plus* 5.1 presents only the Geomin solution with lowest rotation function value.

The main advantage of the ESEM model over existing modeling practices is that it seamlessly incorporates the EFA and SEM models. In most applications with multiple factors the EFA is used to discover and formulate factors. Usually the EFA is followed by an ad-hoc procedure that mimics the EFA factor definitions in an SEM model with a CFA measurement specification. The ESEM model accomplishes this task in a one-step approach and thus it is a simpler approach. In addition, the ESEM approach is more accurate because it avoids potential pitfalls due to the challenging EFA to CFA conversion. For example, an EFA-based CFA model may lead to poor fit when covariates are added to the model. The ESEM approach avoids this problem by estimating the measurement and structural model parts simultaneously.

Many CFA approaches draw on EFA to formulate a simple structure loading specification. The EFA is typically carried out without obtaining standard errors and instead rules of thumb such as ignoring loadings less than 0.3 are used. A CFA based on such an EFA often leads to a misspecified model using chi-square testing of model fit. Model modification searches might not lead to the correct model and fit indexes such as CFA may show sufficiently high values for the model not to be rejected. This article illustrates the possible distortion of estimates that such a CFA-SEM approach can lead to and shows how ESEM avoids the misestimation.

In many modeling applications SEM is used effectively to test substantive theory that is built from considerations unrelated to the data. In such situations the ESEM framework offers an alternative rather than a replacement. If there is a good prior theory then SEM is a valid and simpler approach. However, in real-data examples, especially examples with many measurements and factors, it would be impossible to get the correct loading pattern simply by theoretical considerations. Consider the empirical example discussed in this article. More than half of the measurements presented in Table 1 are of complexity 2. It would be difficult to contemplate this model simply by using substantive theory. A simpler SEM model would provide for a simpler interpretation but would lead to one of three inferior modeling approaches. The first one would ignore the needed cross-loadings, which in turn would lead to biased estimates. The second approach would reject the simple SEM model in favor of a more complicated and more difficult to interpret SEM model perhaps with more factors. The third approach would adjust gradually the initial theoretical model using data-driven results, such as residuals or modification indexes. This third approach, however, is inferior to the ESEM approach because it is essentially an ad-hoc exploratory procedure that resembles manual factor rotation. ESEM provides a theoretically sound alternative based on well-established, optimality-driven rotation criteria.

The ESEM framework can also be used to challenge the conventional wisdom that complexity 1 measurements are important to substantive researchers. One can argue that it is more important to find an accurate set of measurements rather than to find a pure set of measurements. Consider, for example, a simple MIMIC model. One can use an ESEM model to test this theoretical model without worrying about correctly specifying the CFA measurement structure.

ESEM makes possible better model testing sequences. Starting with an EFA measurement specification of only the number of factors, CFA restrictions can be added to the measurement model. Chi-square difference testing can be carried out to study the appropriateness of the CFA restrictions. Previously such testing sequences have been available only outside the SEM model structure, but they can now be integrated into SEM.

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For many applications the ESEM model can be considered as a replacement of the more restrictive SEM model. Unlike EFA, which is typically followed by a CFA, the ESEM model does not need to be followed by a SEM model, because it has all of the features and flexibilities of the SEM model. Nevertheless, in certain cases it might be beneficial to follow an ESEM model by a SEM model. For example, in studies with a small sample size, a follow-up SEM model may have more precise estimates because it has fewer parameters. Constructing a follow-up SEM model from a given ESEM model is fairly easy, amounting to fixing at 0 all insignificant loadings. In addition, because the ESEM and SEM models are typically nested, a rigorous test can be conducted to evaluate the restrictions imposed by the SEM model.

The ESEM modeling framework does not limit the researcher's ability to incorporate substantive information in the model. The researcher can use different rotation criteria to reach the factor pattern that most closely represents the substantive thinking, without sacrificing the fit of the model.

This article also discusses the performance of rotation techniques in Monte Carlo studies, showing the advantage of Geomin. Target rotation is shown to provide an approach that bridges EFA and CFA measurement specification.

Longitudinal and multiple-group analysis with EFA measurement structures greatly expands the possibilities of both EFA and SEM. This article illustrates multiple-group analysis in both a real-data and a simulation study.

Another advantage of the ESEM framework is that it easily accommodates EFA simulation studies. Such studies have been rarely published previously. In this new framework EFA simulation studies are as simple as SEM simulation studies. Simulation studies can greatly enhance this research field.

One of the limitations of the ESEM framework is the fact that any structural path between an exploratory factor and another variable can be included in the model only if such a path is included for all exploratory factors from the same exploratory block. There are two reasons for that. First, with a general rotation criteria such as Geomin the exploratory factors are interchangeable and one would not be able to specify a path using an exploratory factor without knowing which factor that is. With the target rotation that is not an issue because the factors are not interchangeable. The second reason is computational. The methodology presented in this article does not provide a way to construct different structural paths for exploratory factors from the same block. Expanding the methodology in that direction would be a valuable future development. Note, however, that this limitation is relatively harmless. If a structural path is needed between an exploratory variable and another variable, simply adding the same structural path for all the exploratory factors in the same block will not harm the model beyond making it less parsimonious. If, indeed, these added structural paths are not needed their estimates will be near zero and would essentially preserve the correct model. Another limitation of the presented methodology is that exploratory factors from the same block cannot be regressed on each other and cannot have a structured variance–covariance matrix such as second-order factor analysis.

The ESEM approach is implemented in M*plus* Version 5.1 and is developed not only for continuous outcomes with maximum likelihood estimation but also for dichotomous, ordered categorical, censored, and combinations of such outcomes with continuous outcomes with limited-information weighted least squares estimation. Other analysis features available include model modification indexes, standardized coefficients and their standard errors, estimation of indirect effects and their standard errors, factor scores, and Monte Carlo simulations.

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APPENDIX A: ADDITIONAL ROTATION CRITERIA

Following is a list of additional rotation criteria implemented in M*plus*.

- CF-Varimax

$$f(\Lambda) = \left(1 - \frac{1}{p}\right) \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l \neq j, l=1}^{m} \lambda_{ij}^2 \lambda_{il}^2 + \frac{1}{p} \sum_{j=1}^{m} \sum_{i=1}^{p} \sum_{l \neq i, l=1}^{p} \lambda_{ij}^2 \lambda_{lj}^2 \tag{A1}$$

For orthogonal rotations this criterion is equivalent to the Varimax criterion

$$f(\Lambda) = -\sum_{j=1}^{m} \left(\sum_{i=1}^{p} \lambda_{ij}^4 - \frac{1}{p} \left(\sum_{i=1}^{p} \lambda_{ij}^2 \right)^2 \right). \tag{A2}$$

- Quartimin/CF-Quartimax

$$f(\Lambda) = \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l \neq j, l=1}^{m} \lambda_{ij}^2 \lambda_{il}^2 \tag{A3}$$

For orthogonal rotations this criterion is equivalent to the Quartimax criterion

$$f(\Lambda) = -\frac{1}{4} \sum_{i=1}^{p} \sum_{j=1}^{m} \lambda_{ij}^4 \tag{A4}$$

- CF-Equamax

$$f(\Lambda) = \frac{2p-m}{2p} \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l \neq j, l=1}^{m} \lambda_{ij}^2 \lambda_{il}^2 + \frac{m}{2p} \sum_{j=1}^{m} \sum_{i=1}^{p} \sum_{l \neq i, l=1}^{p} \lambda_{ij}^2 \lambda_{lj}^2 \tag{A5}$$

- CF-Parsimax

$$f(\Lambda) = \frac{p-1}{p+m-2} \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l \neq j, l=1}^{m} \lambda_{ij}^2 \lambda_{il}^2 + \frac{m-1}{p+m-2} \sum_{j=1}^{m} \sum_{i=1}^{p} \sum_{l \neq i, l=1}^{p} \lambda_{ij}^2 \lambda_{lj}^2 \tag{A6}$$

- CF-Facparsim, Factor Parsimony

$$f(\Lambda) = \sum_{j=1}^{m} \sum_{i=1}^{p} \sum_{l \neq i, l=1}^{p} \lambda_{ij}^2 \lambda_{lj}^2 \tag{A7}$$

- Crawfer, Crawford-Ferguson family

$$f(\Lambda) = (1-k) \sum_{i=1}^{p} \sum_{j=1}^{m} \sum_{l \neq j, l=1}^{m} \lambda_{ij}^2 \lambda_{il}^2 + k \sum_{j=1}^{m} \sum_{i=1}^{p} \sum_{l \neq i, l=1}^{p} \lambda_{ij}^2 \lambda_{lj}^2 \tag{A8}$$

  where k is a parameter.
- Oblimin

$$f(\Lambda) = \sum_{j=1}^{m} \sum_{l \neq j, l=1}^{m} \left(p \sum_{i=1}^{p} \lambda_{ij}^2 \lambda_{il}^2 - k \sum_{i=1}^{p} \lambda_{ij}^2 \sum_{i=1}^{p} \lambda_{il}^2) \right) \tag{A9}$$

  where k is the parameter.

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APPENDIX B: ROW STANDARDIZATION

Typically the optimal rotation is determined by minimizing the rotation criteria using the standardized loadings, that is, the loadings standardized to correlation scale as in Equations 10 and 11. An alternative standardization frequently used in practice is the Kaiser standardization. In that case the optimal rotation is determined by minimizing the rotation criteria

$$f(D_d^{-1} \Lambda H^{-1}) \tag{B1}$$

over all oblique or orthogonal matrices H where

$$D_d = \sqrt{diag(\Lambda \Lambda^T)} \tag{B2}$$

Another alternative approach implemented in M*plus* is to determine the optimal rotation by using the raw loadings matrix, using the original scales of the variables. In that case

$$f(\Lambda H^{-1}) \tag{B3}$$

is minimized over all oblique or orthogonal matrices H.[17]

APPENDIX C: EFA STANDARD ERRORS

The asymptotic distribution of the rotated solution is based on the following general fit function method. Suppose that S_0 is a correlation matrix and Σ_0 is the estimated correlation matrix, based on an EFA model. Let $F(S_0, \Sigma_0)$ be a general fit function that is minimized to obtain the EFA parameters Λ and Ψ under the rotation constraints Equation 7 or Equation 9, and denote these constraint equations by R. Two examples of such functions are the likelihood fit function

$$F(S_0, \Sigma_0) = ln(|\Sigma_0|) + Tr(\Sigma_0^{-1} S_0) \tag{C1}$$

and the least squares fit function

$$F(S_0, \Sigma_0) = \sum_{i<j} (\sigma_{0ij} - s_{0ij})^2. \tag{C2}$$

It is possible to obtain the asymptotic distribution of the rotated solutions using the asymptotic distribution of S_0. Using the Lagrange multipliers method the rotated solution is also the local extremum for the augmented function

$$F_1(S_0, \Sigma_0) = F(S_0, \Sigma_0) + L^T R \tag{C3}$$

where L is a vector of new parameters. The asymptotic distribution for the parameters that minimize the new fit function is obtained, see Theorem 4.1 in Amemiya (1985), by the sandwich estimator

$$\left(\frac{\partial^2 F_1}{(\partial(\theta, L))^2} \right)^{-1} Var \left(\frac{\partial F_1}{\partial(\theta, L)} \right) \left(\frac{\partial^2 F_1}{(\partial(\theta, L))^2} \right)^{-1} \tag{C4}$$

where the second derivative with respect to the model parameters and the new parameters L is given by

$$\frac{\partial^2 F_1}{(\partial(\theta, L))^2} = \begin{pmatrix} \frac{\partial^2 F}{(\partial\theta)^2} & \frac{\partial R}{\partial\theta} \\ \frac{\partial R}{\partial\theta} & 0 \end{pmatrix}. \tag{C5}$$

[17] The standardization option is controlled in M*plus* by the *RowStandardization=* command and the three options described earlier are *RowStandardization= Correlation, Kaiser,* or *Covariance*.

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The preceding matrix is called the bordered information matrix when the fit function is the likelihood fit function. In fact, the inverse of that matrix alone can be used as an estimator of the asymptotic distribution of the maximum likelihood estimates. The middle term in Equation C4 is the variance of the score and is computed as follows:

$$Var\left(\frac{\partial F_1}{\partial(\theta, L)}\right) = \frac{\partial^2 F_1}{\partial\theta\partial S_0} Var(S_0) \left(\frac{\partial^2 F_1}{\partial\theta\partial S_0}\right)^T \tag{C6}$$

where θ is the vector of model parameters and

$$\frac{\partial^2 F_1}{\partial\theta\partial S_0} = \begin{pmatrix} \frac{\partial^2 F}{\partial\theta\partial S_0} \\ 0 \end{pmatrix}. \tag{C7}$$

The general fit function method described earlier is utilized in ESEM as follows. Using the asymptotic distribution of the unrotated solution, the asymptotic distribution of the estimated correlation matrix is computed via the delta method. The asymptotic distribution of the rotated solution is then obtained from the general fit function method by substituting the estimated correlation matrix for S_0 earlier and using either the Equation C1 or C2 fit functions. Because the fit of the model is perfect, both fit functions lead to the same result.

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APPENDIX D: SIMULATION STUDIES WITH ESEM AND EFA

In ESEM, as well as EFA, the order of all factors is interchangeable and each factor is interchangeable with its negative. These indeterminacies are typically not important. However, they are important in simulation studies where accumulations across the different replications are done to evaluate MSE, parameter estimates bias, and confidence interval coverage.

To avoid this problem additional parameter constraints are used. For example, to identify a factor over its negative the following restriction on the loadings is incorporated

$$\sum_i \lambda_{ij} > 0. \tag{D1}$$

In addition, to make sure that the factors appear consistently in the same order across the replications the following quantities are computed:

$$d_j = \frac{\textit{average index of the large loadings}}{\sum_i \lambda_{ij}^2} \tag{D2}$$

where the large loadings are the loadings that are at least 0.8 of the largest loading. For example suppose that the loadings of a factor are (0.2, 1, 0.9, 0.9, 0, 0.1). The large loadings are loadings 2, 3, and 4, and therefore the average index of the large loadings is 3. The factors are ordered so that

$$d_1 < d_2 < ... < d_m. \tag{D3}$$

This rule guarantees that factors with large loadings on the first dependent variables will tend to appear first.[18] In addition, factors that explain more of the dependent variables' covariance matrix will appear first. This is the effect of the denominator in the definition of d_j.

Simulation studies that are presented here are constructed in a way that ensures that the order of the factors is the same across the replications as well as the sign of the factors. The constraints in Equations D1 and D3, however, will not work for any simulation study and a different set of constraints might have to be used to ensure stable factor order and factor signs. Simulation studies that do not include proper constraints similar to Equations D1 and D3 will lead to meaningless results as they will combine factor loadings from different factors across the replications. Such simulation studies will not give good results and will not provide any information for the quality of the estimation method. Parameter constraints D1 and D3 are important only for simulation studies. These constraints have no implication for a single replication analysis such as real data analysis. It is well known that the order of the factor is exchangeable and that each factor can be replaced with its negative. Because the data do not contain any information about the order of the factors or their signs, it is up to the analyst to make that choice.[19]

A new alignment method is implemented in M*plus* Version 5.2. This alignment method utilizes the starting values provided by the user. The starting values are actually not used during the optimization routine but are used as true parameter values to compute the coverage probabilities for the estimated confidence limits. Denote these starting values as λ_{0ij} The new alignment criteria minimizes the target function

$$\sum_{i,j}(\lambda_{0ij} - s_j \lambda_{i\sigma(j)})^2$$

over all factor permutations σ and sign assignments $s_j = 1$ or -1. Thus the solution that is selected is the one that is the closest to the starting value in the least squares metric.

APPENDIX E: M*plus* INPUT

Following is the M*plus* input for the small cross-loadings simulation study presented in this article. Comment lines begin with (!) and are provided here only for clarity. They are not needed in general.

! this section specifies the simulation framework montecarlo:

names = y1-y10 x;
nobs = 1000;
nreps = 100;

[18]In simulation studies for SEM models M*plus* uses user-specified starting values to ensure that the order of the factors is the same across the replications. However, ESEM and EFA analysis in M*plus* do not use user-specified starting values.

[19]M*plus* will use the constraints in Equations D1 and D3 even for real data analysis, so the factors and their signs are always uniquely determined by M*plus*.

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! this section specifies the parameters for the data generation model population:

```
[x@0]; x@1;
f1 by y1-y5*.8 y6-y10*0;
f2 by y1-y3*0 y4-y5*.25 y6-y10*.8;
y1-y10*.36; [y1-y10*0];
f1-f2@1;
f1 with f2*.5;
f1 on x*.5;
f2 on x*1;
```

! this section specifies the rotation type analysis: rotation = geomin(0.0001);

! this section specifies the model to be estimated and the true
! values to be used for confidence interval coverage rates model:

```
f1 by y1-y5*.8 y6-y10*0 (*1);
f2 by y1-y3*0 y4-y5*.25 y6-y10*.8(*1);
y1-y10*.36; [y1-y10*0];
f1 with f2*.5;
f1 on x*.5;
f2 on x*1;
```

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