Multivariate Behavioral Research

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/hmbr20

Modeling Growth in Latent Variables Using a Piecewise Function

Nidhi Kohli [a] & Jeffrey R. Harring [b]

[a] University of Minnesota

[b] University of Maryland

Published online: 14 Jun 2013.

Multivariate Behavioral Research, 48:370–397, 2013

ISSN: 0027-3171 print/1532-7906 online
DOI: 10.1080/00273171.2013.778191

Routledge
Taylor & Francis Group

Modeling Growth in Latent Variables Using a Piecewise Function

Nidhi Kohli
University of Minnesota

Jeffrey R. Harring
University of Maryland

Latent growth curve models with piecewise functions for continuous repeated measures data have become increasingly popular and versatile tools for investigating individual behavior that exhibits distinct phases of development in observed variables. As an extension of this framework, this research study considers a piecewise function for describing segmented change of a latent construct over time where the latent construct is itself measured by multiple indicators gathered at each measurement occasion. The time of transition from one phase to another is not known a priori and thus is a parameter to be estimated. Utility of the model is highlighted in 2 ways. First, a small Monte Carlo simulation is executed to show the ability of the model to recover true (known) growth parameters, including the location of the point of transition (or knot), under different manipulated conditions. Second, an empirical example using longitudinal reading data is fitted via maximum likelihood and results discussed. Mplus (Version 6.1) code is provided in Appendix C to aid in making this class of models accessible to practitioners.

Increasingly, investigators are interested in how variables capturing facets of cognitive or behavioral development change across a particular span of time. Initially, the scientific objective in these studies focuses on effectively describing patterns of change for each individual as well as the population mean trajectory. Measuring change over time necessarily requires a longitudinal perspective

Correspondence concerning this article should be addressed to Jeffrey R. Harring, Measurement, Statistics & Evaluation, 1230 Benjamin Building, University of Maryland, College Park, MD 20742-1115. E-mail: harring@umd.edu

where repeated measurements are gathered for a collection of individual participants. The latent growth curve (LGC) model (Meredith & Tisak, 1990), a special subclass within structural equation modeling (SEM), is commonly employed to analyze continuous repeated measures data of this type. Theoretically, this approach posits the existence of latent trajectories capturing an underlying process that can only be observed indirectly via the repeated measures (Bollen & Curran, 2006). For instance, researchers in cognitive development may theorize the presence of an unobserved ability to read that develops as a continuous function of time. The repeated measures enable the estimation of the underlying reading ability trajectories that give rise to the measures over time. It is this trajectory estimate that is of primary interest in subsequent modeling.

In addition to finding a functional form that summarizes the repeated measures satisfactorily, assessing patterns of variability stemming from (a) between-subject heterogeneity and (b) within-subject fluctuations is critical to properly specifying the LGC model. The LGC model allows the correlational structure of the repeated measures to be disentangled into intraindividual (within-person) variability as well as interindividual (between-person) variability in individual participants' growth characteristics across time (Preacher, Wichman, MacCallum, & Briggs, 2008). Furthermore, the covariance matrices that model these patterns of between-subjects and within-subjects variability can be tailored to account for interesting features of the data or longitudinal design (see, e.g., Fitzmaurice, Laird, & Ware, 2011; Verbeke & Molenberghs, 2000).

A classic application of LGC models specifies a function describing a linear change process often comprised of two latent growth factors: (a) an intercept that describes initial level or status at some temporal reference point and (b) a linear slope of growth that summarizes constant change over time. These two latent growth factors can be characterized by their mean values, individual random variation, and covariation around these two latent growth components (Duncan, Duncan, & Strycker, 2006). Certainly, other functional forms besides one that posits a linear change process for the repeated measures are possible. In lieu of choosing a model on a strictly theoretical basis, summarizing the repeated measures data in this way is typically accomplished via an empirical exploration of the data. For repeated measures data that exhibit curvilinear behavior, the LGC framework is flexible enough to accommodate a variety of nonlinear functions (see, e.g., Choi, Harring, & Hancock, 2009; Grimm & Ram, 2009). For example, a quadratic function may be proposed for a developmental process that increases across initial measurement occasions, attains some maximum level of performance or proficiency, and finally declines toward the end of study period—perhaps due to fatigue or some other phenomenon. In other research scenarios, individual performance on a learning task that levels off toward the end of the study period may suggest choosing an intrinsically nonlinear function that can represent this type of limiting, asymptotic behavior. Another possibility allows the functional form for the repeated measures not to be specified in advance but rather to be estimated (see, e.g., Meredith & Tisak, 1990). A more detailed discussion of LGC models, along with a number of extensions, can be found in Duncan et al. (2006) as well as Preacher et al. (2008).

An LGC model that examines change across time in repeated measurements of observed variables is termed a "first-order" LGC model. An extension of first-order LGC models are "second-order" LGC models that describe change in a latent construct over time, where the latent construct of interest is measured by multiple indicators gathered at each measurement occasion. In second-order LGC models the first-order latent factors are modeled as dependent on one or more second-order latent growth factors, with the latter having only the first-order latent factors as indicator variables. Thus, second-order latent factors explain the means and variances of and covariances among first-order latent factors (see, e.g., Duncan et al., 2006; Hancock, Kuo, & Lawrence, 2001). Of course, auxiliary variables representing individual attributes, demographic information, or treatment condition can be incorporated to explain why second-order latent growth characteristics differ among individuals. This parallels many applications of first-order LGC models in which investigating treatment effectiveness or attributing differences in growth characteristics to subject-specific explanatory variables is accomplished at a secondary stage of the analysis—typically after the functional form of the repeated measures has been established.

Whether first-order or second-order LGC frameworks are used to investigate longitudinal change, the vast majority of research studies using LGC models regularly presume that the functional form describing the overall change process in the repeated measures data is a smooth, continuous curve with no breaks, elbows, or other irregularities. However, assuming a single uninterrupted functional form underlies the overall change process may be unrealistic for applications where data are comprised of different growth phases. Piecewise latent growth curve (PLGC) models, an extension of LGC models, allow the incorporation of separate growth profiles corresponding to multiple developmental stages from which repeated observations are made (Chou, Yang, Pentz, & Hser, 2004). PLGC models are flexible because each phase can be specified to conform to a particular functional form of the overall change process (Cudeck & Harring, 2010). The term *piecewise* originates from a piecewise regression model, which is a special case of a spline regression model (Marsh & Cormier, 2001). To make this idea more concrete, consider a linear-linear piecewise process. In this situation, the formulated model assumes a simple regression line for the dependent variable but with possibly different parameterizations in different ranges of the predictor (Bates & Watts, 1988; see also Seber & Wild, 1989, Chapter 9). Figure 1 shows a plot of a linear-linear process.

One of the most interesting features of a piecewise model is the knot or changepoint. The knot is the value of the predictor where the "pieces" from the

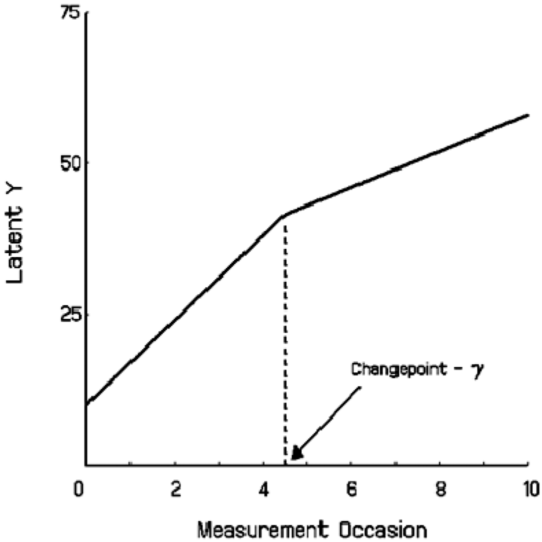

FIGURE 1 Plot of generic linear-linear process for a latent dependent variable Y, with knot at γ.

developmental stages meet and can be known a priori (fixed) or freely estimated. The knot is denoted as γ in Figure 1. Harring, Cudeck, and du Toit (2006) demonstrated how a first-order piecewise linear mixed effects model, where the location of the knot was unknown, could be fit as an SEM to data for investigating individual behavior that exhibited distinct phases in observed variables.

The purpose of the current study is to extend the first-order PLGC model to a second-order structure to examine a linear-linear piecewise change process in latent variables, where the location of knot is unknown. The proposed model is a hybrid of the second-order PLGC model with “fixed” knot location described by Sayer and Cumsille (2001) and the first-order PLGC model with “unknown” knot location described by Harring et al. (2006). That is, the proposed model permits change in a latent variable at the individual level but whose point of transition from one phase to another is unknown a priori. This particular proposed model can be quite useful and relevant, especially in education, psychology, and developmental studies, as so many developmental processes, such as the acquisition of foundational vocabulary knowledge (see empirical example in the later section), may progress in two phases where the functional form in each phase may well be different. The time at which the trajectory for behavior transitions from one phase to the other (i.e., the knot) is important scientifically and often marks a substantive watershed moment (e.g., a level of proficiency has been attained) or suggests when an intervention may be most beneficial.

In the context of the proposed model, the latent variable is measured by the same multiple indicators gathered at each measurement occasion, although this restriction is not necessary to draw valid longitudinal inferences (see, e.g., Bollen & Curran, 2006; Hancock & Buehl, 2008). Although the knot is to be estimated, it is assumed to be the same across individuals. At first glance, constraining the knot to be the same across individuals may seem overrestrictive, yet in many biological or behavioral processes it does not seem unreasonable that some watershed event may occur at roughly the same moment in time for all individuals. For instance, in reading research it is hypothesized that fluency, a measure of accurate and automatic decoding at an appropriate pace, may increase at one rate beginning in second grade but then changes at a different, slower rate for most students in the middle of their third-grade year (R. Silverman, personal communication, June 15, 2011). If grade is used as a proxy for the timing of collected observations, the transition between two phases of fluency development might be expected to be the same for all students but unknown a priori. Because the knot enters the function in a nonlinear manner but is fixed across individuals; this second-order PLGC model turns out not to be much more complicated to set up than a restricted factor analysis with structured mean vector and covariance matrix (see, e.g., Blozis, 2006; Harring, Kohli, Silverman, & Speece, 2012). Thus, SEM software—with all of its features—can be utilized as the platform for estimating model parameters. The estimation of this model is carried out in Mplus 6.1 (Muthén & Muthén, 1998–2010), a popular SEM program. Mplus code for the model can be found in Appendix C.

The remainder of the article is outlined in the following way: In the next section, the model is developed and the likelihood function specified. In the subsequent section, a small Monte Carlo simulation is performed to empirically investigate the ability of second-order PLGC models to recover true (known) growth parameters. Specifically, the current research compared the performance of the second-order PLGC model under different manipulated factors of (a) sample size, (b) location of the knot, and (c) reliability of indicator variables. Reading data obtained from a longitudinal study is introduced and analyzed in the next section. Finally, conclusions are framed in terms of the model's limitations as well as directions for future research.

MODEL SPECIFICATION

Measurement Model

In a second-order PLGC model the repeated measure to be analyzed is an unobservable construct; hence to fit this model to data the model is augmented to include a measurement model that directly connects the observed variables to the latent factors. This relation is typically operationalized in terms of a measurement model connecting the observed indicators with the corresponding latent variable across time. Consider the $(k \times 1)$ response vector, $\mathbf{y}_{ij} = (y_{ij1}, y_{ij2}, \ldots, y_{ijk})'$ for individual i, $i = 1, \ldots, n$, at time j with $1 \leq j \leq m$. It is assumed that these k observed variables at time j measure a single latent variable, η_i, for the ith individual. A linear factor model (cf. Lawley & Maxwell, 1971) is specified that characterizes the relation of the observed variables to the latent variable:

$$\mathbf{y}_{ij} = \boldsymbol{\mu}_j + \boldsymbol{\lambda}_j \boldsymbol{\eta}_{ij} + \boldsymbol{\delta}_{ij}, \tag{1}$$

where $\boldsymbol{\mu}_j$ is a $k \times 1$ vector of variable intercepts, $\boldsymbol{\lambda}_j$ is a $k \times 1$ vector of fixed or unknown factor loadings that describe the linear relation between the latent variable and the manifest variables, $\boldsymbol{\eta}_{ij}$ is the latent variable, and $\boldsymbol{\delta}_{ij}$ is a $k \times 1$ vector of time-specific unique factors. Like standard factor analysis, the common factor is assumed to be independent of the errors (i.e., $\mathrm{cov}(\boldsymbol{\delta}_{ij}, \boldsymbol{\eta}_{ij}) = \mathbf{0}$). Furthermore, once the linear dependence among the manifest variables is accounted for, the unique factors are assumed to be mutually independent (i.e., $\mathrm{cov}(\boldsymbol{\delta}_{ij}, \boldsymbol{\delta}_{ij'}) = 0$, for all $j \neq j'$).

In many situations where multiple instruments are used in a longitudinal design, it is not unusual for the same battery to be given repeatedly. This is the situation found in the subsequent empirical example. In this case, if a complete set of the same k variables were obtained at multiple occasions—with a maximum of m potential time points ($j = 1, \ldots, m$) then individual i would have a response vector with a total number of $T = mk$ observations—although other design considerations are certainly possible depending on the availability of the same instrumentation (Bollen & Curran, 2006) and whether or not the indicators of the construct shifts over time (Hancock & Buehl, 2008).

Working from the scenario that the observed variable indicators are identical at each time point, let $\mathbf{y}_i' = (\mathbf{y}_{i1}', \ldots, \mathbf{y}_{im}')$ denote a $T \times 1$ vector of responses for individual i, stacked according to j across all m occasions. Similarly, the linear factor model can be viewed as the stacked response vectors across all m measurement occasions can be specified as

$$\mathbf{y}_i = \boldsymbol{\mu} + \boldsymbol{\Lambda} \boldsymbol{\eta}_i + \boldsymbol{\delta}_i. \tag{2}$$

In Equation 2, $\boldsymbol{\mu}$ is a $T \times 1$ vector of intercepts,

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_m \end{bmatrix},$$

$\mathbf{\Lambda}$ is a $T \times m$ block diagonal matrix of factor loadings,

$$\mathbf{\Lambda} = \begin{bmatrix} \boldsymbol{\lambda}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \boldsymbol{\lambda}_m \end{bmatrix},$$

$\boldsymbol{\eta}_i$ is an $m \times 1$ vector of latent factors corresponding to individual i,

$$\boldsymbol{\eta}_i = \begin{bmatrix} \eta_{i1} \\ \vdots \\ \eta_{im} \end{bmatrix},$$

and $\boldsymbol{\delta}_i$ is a $T \times 1$ vector of unique factors,

$$\boldsymbol{\delta}_i = \begin{bmatrix} \boldsymbol{\delta}_{i1} \\ \vdots \\ \boldsymbol{\delta}_{im} \end{bmatrix}.$$

The distribution of the unique factors is given as

$$\boldsymbol{\delta}_i \sim N(\mathbf{0}, \boldsymbol{\Theta}(\boldsymbol{\varphi})). \tag{3}$$

The matrix $\boldsymbol{\Theta}(\boldsymbol{\varphi})$ is a $T \times T$ symmetric covariance matrix in which the diagonal elements contain the variances of the unique factors corresponding to the linear factor model of the repeated measures whereas the off-diagonal elements are their corresponding covariances. Unlike conventional factor analysis where the covariance matrix of the unique factors is assumed to be strictly diagonal, specification of off-diagonal elements of $\boldsymbol{\Theta}(\boldsymbol{\varphi})$ under the longitudinal design implied in Equation 2 is commonplace. For example, allowing covariances of temporally adjacent pairs of unique factors to be freely estimated would seem plausible given that the same indicators are measured repeatedly over time. In some domains, the within-individual variances may actually increase or decrease systematically—a situation in which variances may depend on the mean. Other structures can be tailored to correspond with other design, theoretical, or empirical considerations with the stipulation that this be done as parsimoniously as possible.

Piecewise Model for the Latent Repeated Measures

The structural model for the repeated latent variable is a two-phase linear-linear latent growth process with a piecewise function:

$$\boldsymbol{\eta}_i = g_{ij}(t_j, \boldsymbol{\alpha}_i, \gamma) + \boldsymbol{\zeta}_i, \tag{4}$$

where $\boldsymbol{\zeta}_i$ is a vector of random disturbances in the first-order latent factors, $\boldsymbol{\eta}_i$, that are often assumed to be normally distributed with mean vector, $\mathbf{0}$, and covariance matrix $\boldsymbol{\Delta}$ (i.e., $\boldsymbol{\zeta}_i \sim N(\mathbf{0}, \boldsymbol{\Delta})$) and uncorrelated with $\boldsymbol{\alpha}_i$ and $\boldsymbol{\delta}_i$. Function g defines a general piecewise regression model. For repeated latent variables that may follow a linear-linear trend, g is specified as

$$g_{ij} = \begin{cases} \alpha_{i1} + \alpha_{i2} t_j & t_j \leq \gamma \\ \alpha_{i3} + \alpha_{i4} t_j & t_j > \gamma \end{cases}, \tag{5}$$

where t_j is the jth time point, γ is the unknown knot, α_{i1} and α_{i2} are the intercept and linear slope of the first segment, and α_{i3} and α_{i4} are the intercept and linear slope of the second segment. Note that the regression coefficients have an i subscript and therefore vary by individual whereas the knot, γ, is fixed for all participants. Although not universally true, if it is presumed that the functions characterizing the two phases join at the knot, then the function values at γ are equal (i.e., $\alpha_{i1} + \alpha_{i2}\gamma = \alpha_{i3} + \alpha_{i4}\gamma$). This implies that one parameter is unnecessary and can be eliminated. Of the four regression parameters, α_{i3} seems the least interesting as it corresponds to the value of η at $t = 0$ of the second segment—a point not pertinent to the second phase. In the end, the choice is completely arbitrary. The terms of the equality constraint can be rearranged and solved for α_{i3}: $\alpha_{i3} = \alpha_{i1} + \alpha_{i2}\gamma - \alpha_{i4}\gamma$. Equation 6 shows this modification:

$$g_{ij} = \begin{cases} \alpha_{i1} + \alpha_{i2} t_j & t_j \leq \gamma \\ \alpha_{i1} + \alpha_{i2}\gamma + \alpha_{i4}(t_j - \gamma) & t_j > \gamma \end{cases}. \tag{6}$$

The number of parameters that must be estimated in the Equation 6 is four, three linear coefficients: $\boldsymbol{\alpha}_i' = (\alpha_{i1}, \alpha_{i2}, \alpha_{i4})$ and one nonlinear coefficient, γ. Note that it is not necessary for the two segments to always join at the knot, γ. In this scenario, the model specification will be identical to that in Equation 5. Cudeck and Codd (2012), for example, proposed a piecewise function for modeling a phenomena called *reminiscence* (i.e., the demonstration of improved memory or learning without practice or review) in which the reminiscence effect was formulated as the change in the response—a target score on the pursuit rotor task—over repeated trials on successive days. This effect leads to a piecewise function whose segments are disjointed across days.

As a starting point, an individual's regression coefficients, $\boldsymbol{\alpha}_i$, are simply the sum of fixed and random effects

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha} + \mathbf{a}_i,$$

where the random effects are assumed to be multivariate normal such that $\mathbf{a}_i \sim N(\mathbf{0}, \mathbf{\Phi})$. Time-invariant covariates, $\mathbf{x}_i$, for participant i can be incorporated in a straightforward manner as

$$\boldsymbol{\alpha}_i = \boldsymbol{\alpha} + \mathbf{\Gamma}_x \mathbf{x}_i + \mathbf{a}_i,$$

where $\mathbf{\Gamma}_x$ is a matrix of regression coefficients relating the covariates to the growth parameters.

In its current form, the model in Equation 6 together with the measurement model in Equation 2 cannot be directly estimated within SEM software. The difficulty stems from the inability of the software to incorporate executable programming functions, like if-then statements, in the estimation step. In other environments, there have been several solutions put forth to work around this problem including using built-in minimum/maximum functions or user-defined programmable statements within the statistical software module. A parameterization used here was first introduced by Harring et al. (2006), which circumvents this problem by rewriting the function as a polynomial and using the nonlinear constraints feature now pervasive in most SEM software packages. Appendix A illustrates the parameterization procedure used in this study. Following Harring et al. (2006), the reparameterized model is

$$g_{ij} = \beta_{i1} + \beta_{i2} t_j + \beta_{i3} \sqrt{(t_j - \gamma)^2}. \tag{7}$$

In Equation 7, the original coefficients in Equation 6 are reparameterized as follows: $\beta_{i1} = (\alpha_{i1} + \alpha_{i3})/2$, $\beta_{i2} = (\alpha_{i2} + \alpha_{i4})/2$ and $\beta_{i3} = (\alpha_{i4} - \alpha_{i2})/2$. The newly formed parameters $\boldsymbol{\beta}_i' = (\beta_{i1}, \beta_{i2}, \beta_{i3})$ are assumed to follow a multivariate normal distribution $\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \mathbf{\Omega})$. Upon convergence of the program, the estimated original regression coefficients and their corresponding standard errors can be reconstructed via the multivariate delta method (Oehlert, 1992). The procedure of transforming the estimated regression coefficients back to the original function is described in Appendix B. Note that the estimated location of knot comes out as a function of time; hence, it does not require any kind of transformation.

MAXIMUM LIKELIHOOD ESTIMATION

All of the parameters on the right side of Equation 7 that have i subscripts enter function g in a linear fashion. Thus, the model in Equation 7 can be written in matrix form as $g_{ij} = \mathbf{\Gamma}_\beta(\gamma)\boldsymbol{\beta}_i$. The coefficient matrix $\mathbf{\Gamma}_\beta(\gamma)$ is a function of constants, time, and nonlinear parameter γ with the jth row of $\mathbf{\Gamma}_\beta(\gamma)$ defined as

$$\mathbf{\Gamma}_\beta(\gamma)_j = [1 \quad t_j \quad \sqrt{(t_j - \gamma)^2}].$$

In the final staging of formulating the model, the first-order linear factor model for manifest variables in Equation 2 and the model for the first-order factors in Equation 4 with the model for g in Equation 7 substituted in Equation 4 can be expressed jointly as follows:

$$\mathbf{y}_i = \boldsymbol{\mu} + \boldsymbol{\Lambda}[\boldsymbol{\Gamma}(\gamma)\boldsymbol{\beta}_i + \boldsymbol{\zeta}_i] + \boldsymbol{\delta}_i.$$

Given the distributional assumptions of $\boldsymbol{\beta}_i$, $\boldsymbol{\zeta}_i$, and $\boldsymbol{\delta}_i$ the model-implied mean vector and covariance matrix of the response $\mathbf{y}_i$ are, respectively,

$$\mathrm{E}[\mathbf{y}_i] = \boldsymbol{\mu}_y = \boldsymbol{\mu} + \boldsymbol{\Lambda}\boldsymbol{\Gamma}_\beta(\gamma)\boldsymbol{\beta}$$

$$\mathrm{Var}[\mathbf{y}_i] = \boldsymbol{\Sigma}_y = \boldsymbol{\Lambda}(\boldsymbol{\Gamma}_\beta(\gamma)\boldsymbol{\Omega}\boldsymbol{\Gamma}'_\beta(\gamma) + \boldsymbol{\Delta})\boldsymbol{\Lambda}' + \boldsymbol{\Theta}.$$

Fitting the Model

A second-order PLGC model imposes structures on the mean vector and covariance matrix, $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\theta})$, $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a $z \times 1$ vector whose elements consist of all free parameters of the model. Typically, these models are fitted by minimizing, with respect to $\boldsymbol{\theta}$, a function, $F(\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}(\boldsymbol{\theta}), \boldsymbol{\Sigma}(\boldsymbol{\theta}))$, that measures the discrepancy between the sample mean vector, $\bar{\mathbf{y}}$, and covariance matrix $\mathbf{S}$ and the mean vector and covariance matrix implied by the model, $\boldsymbol{\mu}(\boldsymbol{\theta})$ and $\boldsymbol{\Sigma}(\boldsymbol{\theta})$, respectively. For maximum likelihood estimation the discrepancy function to be minimized is

$$F(\bar{\mathbf{y}}, \mathbf{S}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (\bar{\mathbf{y}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{y}} - \boldsymbol{\mu}) + \ln|\boldsymbol{\Sigma}| - \ln|\mathbf{S}| + tr[(\mathbf{S} - \boldsymbol{\Sigma})\boldsymbol{\Sigma}^{-1}].$$

Estimation of the model requires that the software program is capable of specifying nonlinear constraints. A number of recent articles have explicated their use in fitting a variety of nonlinear growth models (e.g., Grimm & Ram, 2009; Harring et al., 2012; Preacher & Hancock, 2012). Computations were carried out using Mplus 6.1 (see Appendix C for Mplus input file).

SIMULATION STUDY

A Monte Carlo simulation approach was used to investigate the extent to which the performance of a second-order PLGC model was influenced by different population characteristics. Factors that were hypothesized to impact the estimation of the knot, along with the estimation of other model parameters, included sample size, location of the knot, and reliability of indicator variables. To evaluate parameter recovery, the proposed model was fitted to data generated from a population model with true (known) parameters, and parameter estimates were then compared with their true values. The design of the simulation study was a 3 (sample size levels) × 2 (observed variable reliability levels) × 3 (knot location levels) completely crossed factorial design resulting in 18 possible combinations. For each cell, 500 replications were generated to assess parameter recovery. Clearly, the factors chosen in this study hardly represent an exhaustive set. Yet, our primary goal was not to conduct a multifaceted comprehensive empirical study but rather to develop the second-order PLGC model and demonstrate how it may be used in an application with real data. A secondary goal was to provide some guidance to practitioners regarding the feasibility of fitting this model in practice based on results from a small simulation study.

Data Generation

A second-order PLGC model was used as the population model to generate repeated measures data conforming to nine equally spaced time points (coded 0 to 8), following Equation 2, Equation 4, and Equation 6. The data were generated using the R program (R Development Core Team, 2009). It has often been the case in both methodological and substantive research of piecewise growth models that the minimum number of time points is at least six (see, e.g., Cudeck, 1996; Cudeck & Klebe, 2002; Harring et al., 2006), hence, the choice of nine time points seemed to be a reasonable choice. In the data generation process, some factors were fixed throughout all simulations, whereas other factors were manipulated. Both types of factors are subsequently described.

Manipulated Factors

Sample size. Three levels of sample sizes ($n = 100$, $n = 250$, and $n = 500$) were chosen to reflect low, intermediate, and high degrees of estimation precision for measurement and structural parameters, respectively. Previous studies that demonstrated the fit of second-order LGC models to real data used sample sizes ranging from $n = 230$ (Harring et al., 2012), $n = 610$ (Ferrer, Balluerka, & Widaman, 2008), and $n = 1{,}994$ (Blozis, 2006). From preliminary investigations, it was determined that for even more extreme sample sizes (e.g., $n = 750$, $n = 1{,}000$), the increase in estimation precision relative to that observed when $n = 500$ was negligible.

Location of knot. Nine equally spaced repeated measures data were generated according to the sample size condition. The range in which the population values of the knot were chosen was between time point 2 and time point 6. These values were chosen based on the rationale that before time point 2 and after time point 6 there is too little information available to estimate the mean

TABLE 1
Population Values for the Manipulated Conditions

Manipulated Conditions	*Levels*		
	Level 1	*Level 2*	*Level 3*
Sample size	100	250	500
Location of knot	2	4	6
Indicator reliability	0.45	0.85	

Note. The unique factors, δ_{im}, were generated under $\delta_{im} \sim N(0, 1)$ and manipulated with variances of the unique factors chosen so that the reliability of the each indicator corresponded to the two levels chosen for the simulation.

and the variance of the slopes of the first phase or of the slope of the second phase, respectively. Thus, the three levels of location of knot (i.e., $\gamma = 2$, $\gamma = 4$, and $\gamma = 6$) were chosen to reflect the estimation precision for measurement and structural parameters, respectively, when the knot occurs earlier in the process, about midway through the process, and toward the end of the process.

Indicator reliability. An advantage of SEM over latent growth curve models for observed variables is that, by definition, latent variables are measured without random error. Reliability of the observed variable indicators was chosen as a factor to be manipulated because it has been documented to impact estimation of structural parameters in other structural equation models (Dimitruk, Schermelleh-Engel, Kelava, & Moosbrugger, 2007). Based on the literature (see, e.g., Harring et al., 2012; Weiss, 2010), reliabilities of indicator variables were chosen to be 0.45 (indicating poor reliability) and 0.85 (indicating a reasonably good level of reliability). Indicator reliabilities were held equal across the indicator variables, primarily to keep the scope of the simulation manageable although we acknowledge that the reliabilities for each observed variable indicator could very well be distinct in applied research. The manipulated factors are summarized in Table 1.

Population Values

The focal point of this empirical investigation is to examine the feasibility of estimating the location of the knot in a second-order PLGC model under different manipulated conditions that are assumed to impact its estimation. In any simulation, decisions must be made regarding the number of factors to manipulate as well as the number of levels within each factor, knowing full well that not every contingency can be addressed. The population values for the second-order PLGC model were chosen based on the empirical study by

Kohli (2011) as well as on values of parameters resulting from a preliminary examination of the reading data used in the subsequent empirical example. The population generating values are described in Table 2.

Outcome Measures

Upon convergence of the program, the estimated parameters of the reparameterized model were transformed back to the original regression coefficients and their corresponding standard errors using the multivariate delta method. To evaluate the performance of the second-order PLGC model under different manipulated conditions, the following outcome measures were used: relative bias and a variability index of parameter bias. Relative bias and variability index of parameter bias was computed from the parameter estimates $\hat{\theta}_i$ $(i = 1, \ldots, 500)$ obtained from the 500 replications. Bias was computed as the difference between the average of the parameter estimates and the true value of the parameter being estimated. That is,

$$\hat{\theta}_{bias} = 500^{-1} \sum_{i=1}^{500} \hat{\theta}_i - \theta_0,$$

where θ_0 is the population value for θ. Relative bias for each parameter was subsequently defined in relation to its population value.

$$\hat{\theta}_{rel\text{-}bias} = \frac{\hat{\theta}_{bias}}{\theta_0} \times 100\%.$$

Additionally, a variability index for parameter bias corresponding to each of the estimated parameters for each replication in each cell was computed. The

TABLE 2
Population Values for Generating Data for the Second-Order PLGC Model

Parameter	*Value*	*Distribution*
α_1	25	
α_2	5	
α_4	1	
μ	0	
λ	0.7	
$\mathbf{\Phi}$	$\mathbf{\Phi} = \begin{pmatrix} 10 & & \\ 0 & 1 & \\ 0 & 0 & .5 \end{pmatrix}$	$\mathbf{a}_i \sim MVN(\mathbf{0}, \mathbf{\Phi})$
ζ	0.5	$\zeta_i \sim N(0, 1)$

variability index for parameter bias is an index of the stability of parameter estimates. It was computed for each replication for each parameter as

$$\text{Variability Index} = (\hat{\theta}_i - \overline{\hat{\theta}})^2.$$

Furthermore, to quantify relative bias as a function of the manipulated conditions, a completely crossed factorial ANOVA [3 (sample size) × 2 (indicator reliability) × 3 (location of knot)] was executed. Partial eta squared, η_p^2, corresponding to each manipulated factor and the interaction terms, was computed as a method to filter the results in terms of practical importance. Partial η_p^2 for a manipulated factor was defined as the proportion of total variation attributable to the factor, partialling out (excluding) other factors from the total nonerror variation (Pierce, Block, & Aguinis, 2004). Only those main effects of the manipulated factors and any two-way or three-way interaction terms were reported and interpreted when both statistical significance ($p < .05$) and practical significance ($\eta_p^2 \geq 0.06$) were achieved. The latter threshold was chosen to correspond to a medium-size effect (Cohen, 1988).[1]

RESULTS OF SIMULATION STUDY

All 500 replications in each of the 18 manipulated conditions converged successfully, where a properly converged replication was determined to be one in which the solution converged with no parameter estimates outside the possible range for that parameter.

Relative Bias and Variability Index of Bias

The results from the ANOVA analyses indicated that the only manipulated factor that was systematically related to the outcome measure, relative parameter bias with respect to the estimated model parameters, was the location of the knot. Unlike the location of the knot factor, the manipulated factors of sample size and indicator reliability had no associated main effects and/or interaction effects that satisfied both the statistical and the practical significance criteria. The location of the knot was systematically related to the parameter bias with respect to the estimation of the mean of slope of the second segment[2] [$F(2, 8982) = 304.5, p < .001, \eta_p^2 = 0.063$], variance of the mean intercept of the first

[1]Cohen's (1988) heuristic values were computed and reported for η^2. Justification for using the same benchmark for η_p^2 as for η^2 comes from Sapp (2006), who stated that the difference between the two effect-size indices becomes negligible as sample size increases.

[2]There was a small effect size ($\eta_p^2 = 0.02$) for the mean slope of the first segment.

segment $[F(2, 8982) = 5557.0, p < 0.001, \eta_p^2 = 0.553]$, and variance of random disturbances in the first-order latent factors $[F(2, 8982) = 1088.0, p < 0.001, \eta_p^2 = 0.195]$. The subsequent paragraphs further explore the relation between the parameter bias and the location of knot. Furthermore, for the variability index outcome measure, none of the manipulated factors had any associated main effects and/or interaction terms that satisfied both the statistical and the practical significance criteria. Thus, it could be concluded that these manipulated factors, at the particular levels chosen, were not systematically related to the variability index for parameter bias.

Relative bias of the mean slope of the second segment across the three levels of the knot location is pictured in Figure 2. The median of relative bias when the knot was located at $t = 2$ was 1% whereas when $t = 4$ and $t = 6$ were 0% and 4%, respectively. The median was chosen as a measure of location due to the obvious skew induced by several outliers, especially in the $t = 6$ condition. Relative bias in the $t = 6$ condition ranged from −183% to 94%. However, the 95% confidence interval was computed as (−10.1, −8.2), which indicates that the slope of the second segment was underestimated (negative bias) by approximately 10%. On average, the relative bias for the other two conditions was essentially zero. Although not reported due to the partial eta-squared value

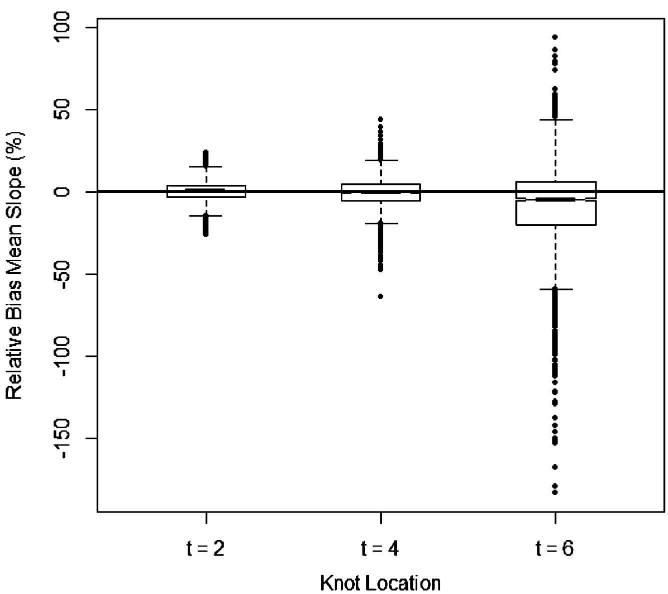

FIGURE 2 Boxplot of relative bias for parameter estimates of the mean slope of the second segment.

not meeting the specified threshold, relative bias for the mean slope of the first segment demonstrated a similar pattern yet in the opposite direction. That is, that the mean slope showed the greatest bias when the knot was located at $t = 2$ and declined to zero at the other conditions.

Relative bias with respect to the variance of the intercept of the first segment was positive and increased as the location of the knot moved from $t = 2$ to $t = 6$. In other words, the variance was increasingly overestimated at all the three levels of the location of knot factor. Graphical representation of relative bias for the variance for the intercept of the first segment using side-by-side boxplots is shown in Figure 3. This finding is not all that surprising because, in general, the estimation of variances/covariances of growth factors, especially in the context of a nonlinear model like a piecewise growth model, is known to be notoriously problematic. Upon closer inspection, when the knot was located earlier in the overall trajectory (i.e., $t = 2$), the formulation of the first segment is quite constrained—having only three time points to define the first linear function. In this case, the variability of the intercept may better coincide with the population value. On the other hand, when the knot is located at $t = 6$, the first segment is comprised of an intercept and slope estimated using more data (and potentially affected by more disturbance error variance—which itself appears to

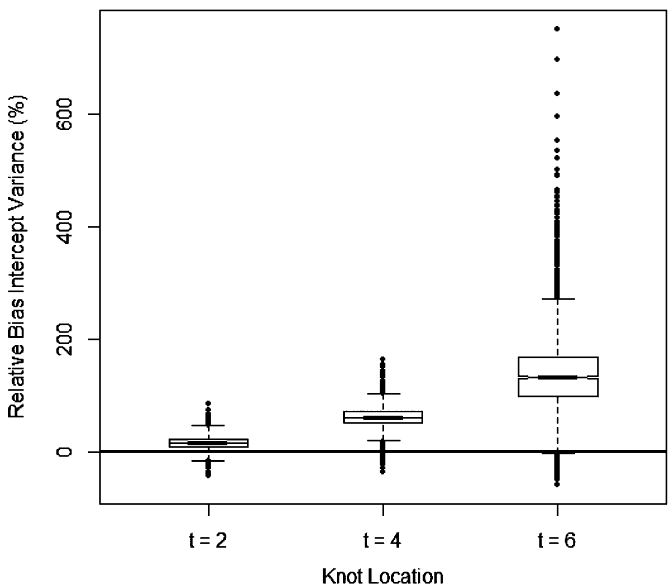

FIGURE 3 Boxplot of relative bias for parameter estimates of the variance of the intercept of the first segment.

have been overestimated). Thus, the intercepts may indeed demonstrate more variability than what was intended via the simulation design. This was certainly an unexpected but interesting finding.

The relative bias for the variance of random disturbances in the first-order latent factors ranged from 30% to 36%. It is clear that the parameter was overestimated at all the three levels of location of knot, although it should be noted that the magnitude of overestimation was not that dissimilar from one another.

AN EXAMPLE: ANALYSIS OF READING DATA

For illustrative purposes, data were collected from a 2-year longitudinal study that investigated the components and processes of vocabulary development among linguistically diverse children as they relate to reading growth over time (Proctor, Silverman, Harring, & Montecillo, 2011). Three hundred ninety-one children participated in the study. There were 144, 130, and 117 participants in Grades 2, 3, and 4, respectively. Fifty-six percent of the sample was comprised of monolingual English speakers. Forty-four percent of the students were Spanish-English bilinguals. Students were recruited from one Northeastern site (n = 121) and one Mid-Atlantic site (n = 270) from one of three schools per site (six total schools). An accelerated cohort-sequential longitudinal design was employed and students were followed over 2 years. Students were assessed in the middle of fall and middle of spring (at approximately half-year increments) during the 2009–2010 and the 2010–2011 school years. A primary objective in the study was to determine a functional form for changes in foundational vocabulary knowledge (depth) as measured by observed language-related process indicators—morphology, syntax, and semantics. Over the span of the study, data were collected at eight time points in half-year increments starting in the fall of second grade and running through the spring of fifth grade.

The latent variable, vocabulary depth, was defined as including observed indicators of morphological awareness, awareness of semantic relations, and syntactic awareness. The Extract the Base test (ETB; Anglin, 1993; August, Kenyon, Malabonga, Louguit, & Caglarcan, 2001; Carlisle, 1988) was individually administered to all students to evaluate awareness of derivational morphology. The score reliability was computed using a Rasch item response theory (IRT) model as 0.98 (August et al., 2001). The Word Classes 2 subtest of the Clinical Evaluation of Language Fundamentals (CELF; Semel, Wiig, & Secord, 2003) was used to measure awareness of semantic relations. Test–retest reliability as indicated in the CELF manual ranges from .83 to .91 for children ages 7.0–9.11. The CELF Formulated Sentences subtest was used to measure this construct. Test–retest reliability as reported in the CELF manual ranged from .74–.79

for children ages 7.0–9.11 and internal consistency as measured by Cronbach's alpha was .80–.82 for these same ages. Raw scores were used in the analysis. The assessments, Extract the Base and Word Classes 2, were conducted only in English whereas the CELF Formulated Sentences subtest was conducted in both English and Spanish. Interested readers are directed to Proctor et al. (2011) for a more comprehensive description of these measures as well as the characteristics of the design and sample.

When using second-order growth models to investigate longitudinal change, an implicit assumption is that the same latent variable has been measured across time. That is, any change in the latent variable is due to true change in the underlying phenomena or construct and not due to changes that may occur in the measurement model. Thus, the invariance of measurement properties of the latent construct over time must be determined in order to draw valid inferences regarding the change process (see, e.g., Ferrer et al., 2008, for a detailed discussion of the issues arising when establishing longitudinal invariance).

Four factor models distinguished by increasingly more stringent levels of factorial invariance were fitted to the data. Model fit was evaluated by assessing and comparing estimates of root mean square error of approximation (RMSEA), the Bayesian information criteria (BIC), and the comparative fit index (CFI). Indices for the fitted models are summarized in Table 3. For the vocabulary depth construct, strong factorial invariance provided reasonable fit to the data. The RMSEA point estimate 0.080 (with 90% confidence interval (CI) at this level of invariance, [0.062, 0.098]), demonstrated fair model fit (Hu & Bentler, 1999). The BIC favored the model with strong factorial invariance as well. The CFI was highest under configural invariance, although values for the model

TABLE 3
Model Comparisons for Varying Levels of Factorial Longitudinal Invariance

Model	*−2lnL*	*p*	*RMSEA*	*RMSEA 90% CI*	*BIC*	*CFI*
1	21,682.6	136	0.057	[0.037, 0.074]	22,494.3	0.983
2	21,724.1	122	0.066	[0.049, 0.082]	22,452.2	0.974
3	21,843.4	108	0.080	[0.062, 0.098]	22,441.2	0.942
4	21,921.9	87	0.102	[0.089, 0.114]	22,488.0	0.924

Note. Model 1 denotes configural invariance. Model 2 represents weak invariance with equal factor loadings, unequal observed variable intercepts, and unequal unique variances. Model 3 denotes strong invariance with equal factor loadings, equal observed variable intercepts, and unequal unique variances. Model 4 represents strict factorial invariance with equal factor loadings, equal observed variable intercepts, and equal unique variances. The number of parameters is p. The baseline model on which CFI is based is a model for which the covariances between temporally adjacent observed variables are nonzero.

under strong invariance still demonstrated relatively good fit (Hu & Bentler, 1999). The fit of model under strict factorial invariance provided the poorer fit compared to the model with strong factorial invariance. Thus, the measurement model under strong measurement invariance was provisionally taken as an acceptable model. This model corresponds to the minimally acceptable level of invariance generally considered necessary for meaningful comparisons of the means (Blozis, 2006).

The data were fit with the measurement model specified in Equation 2 allowing for adjacent time-specific errors for the same observed measure to correlate over time. Growth in latent vocabulary depth followed the model specified in Equation 7. The residuals in Equation 4 were taken to be homogeneous and independent across time (i.e., $\sigma^2\mathbf{I}_{n_i}\,\forall j = 1,\ldots,8$). Through some initial exploration, it was determined that only the intercept of the first phase varied across students whereas the slopes of the first and the second phases were determined to be nonstochastic. This latter point suggests that although students begin at different initial levels of vocabulary depth, they progress at the same rate (albeit different rates in each phase) through fifth grade. Maximum likelihood estimates for the structural model are given in Table 4.

All of the growth parameters were statistically significant at the .05 significance level. Although the growth parameters in Table 4 are on a transformed scale, at first glance it may appear that these are not directly related to the underlying growth of reading depth. Upon closer inspection, the significance test of β_3 is intuitively appealing. Recall that $\beta_3 = (\alpha_4 - \alpha_2)/2$, and thus a test of the null hypothesis ($H_0 : \beta_3 = 0$) answers the question of whether or not a

TABLE 4
Maximum Likelihood Estimates of the Linear-Linear Growth Model of Latent Reading Depth

Parameter	*Estimate*	*SE*
Change characteristics[a]		
β_1	27.73	0.72
β_2	5.77	0.47
β_3	−2.05	0.49
γ	1.41	0.15
Change characteristics variances and covariance matrix[b]		
$\text{var}(\beta_{1i}) = 62.2$		

[a]The parameter estimates are of the transformed model described in Equation 7. [b]The variance of the random effect for β_2 and β_3 was determined to be zero. A subsequent model was fitted that allowed only variance components of random effects for β_1 to be estimated. The lone variance parameter was statistically significant at the .05 level.

two-segmented process is operating or simply a single linear process (Seber & Wild, 1989). The parameter estimate compared with its standard error revealed that a second segment may very well be functioning in the population. Any further interpretation with the transformed coefficients is not essential.

Making sense of the original coefficients requires that we back transform to the original parameters (and their standard errors), which represent growth characteristics that are more interpretable in terms of summarizing the developmental trajectory of vocabulary depth. As was mentioned before, this requires the multivariate delta method (see Appendix B for details).

The knot, signaling the transition from one phase to the other, was estimated to occur at $t = 1.41$. Time was scaled so that the initial value ($t = 0$) occurred at the fall of second grade. Approximately one and a half years later, in the spring of third grade, students' growth in vocabulary depth slows down. This finding aligns with the theoretical notion that most growth in vocabulary depth occurs in the early grades and decreases as schoolchildren move through the upper primary grades. Following Harring et al. (2006), the coefficients from the original spline can be reconstructed from β_1, β_2, β_3, and γ as follows:

$$\alpha_1 = \beta_1 + \beta_3\gamma \qquad \alpha_2 = \beta_2 - \beta_3 \qquad \alpha_3 = \beta_1 - \beta_3\gamma \qquad \alpha_4 = \beta_2 + \beta_3.$$

These back-transformed coefficients and their corresponding standard errors are provided in Table 5. The intercept and slope of the first phase were computed to be 24.84 and 7.82, respectively. This means that on average, students grow approximately 5 raw scale score points (on the ETB scale) for each half-year increment until the spring of the third grade at which time they progress at a slower rate of 3.72 raw scale score points per half-year until the spring of fifth grade. It is interesting to note that although the intercept of the second phase (e.g., α_3) was originally eliminated to force the segments to join at the knot, it is still possible to get an estimate for this value ($\hat{\alpha}_3 = 30.62$).

TABLE 5
The Back-Transformed Original Coefficients in Equation 6 Computed From the Multivariate Delta Transformation

Parameter	*Estimate*	*SE*
Change characteristic		
α_1	24.84	1.08
α_2	7.82	0.89
α_4	3.72	0.36

DISCUSSION

This research study considers a second-order PLGC model with unknown knot location for describing segmented change in a latent construct across time, where the latent construct is measured by a set of observed variables at each time occasion. Formulation of this model requires specifying a measurement model that directly connects the observed variables to the latent factors is augmented to the structural portion of the model. Both the measurement and the structural portion of the model in Equation 2 and Equation 6, respectively, cannot be directly estimated within SEM software, however. Hence, to fit this segment model, the original model in Equation 6 needs to be reparameterized. An obvious limitation of reparameterization is that the fit of the model may be affected by the transformation from one version of a model into another form. Harring et al. (2006) mentioned that generally the difference in fit is not great, and any slight loss in fit would seem to be offset by the ease with which the reparameterized model can be estimated.

A small simulation study was conducted to determine if a second-order PLGC model could be fit and if population parameters, especially those associated with the latent growth process, could be recovered. Unsurprisingly, the location of knot affected the accuracy, in terms of bias, of the mean slope of the second phase. When the shift from one transition to the other occurs at later measurement occasions, the slope of the second phase was negatively biased. The location of knot also affected the variability of the first-phase intercepts. The results showed that the bias (positive) in variability increased as the knot moved farther away from the intercept. Of course, we examined only a few conditions with a finite number of levels of factors thought to impact the precise and accurate estimation of model parameters. The simulation size notwithstanding, the results were encouraging overall.

To further demonstrate the efficacy of the method, reading data collected from a cohort-sequential longitudinal design was analyzed. For this sample, latent reading depth developed in two phases from second grade to fifth grade. The first phase was characterized by faster straight-line growth until the latter half of third grade at which time growth in reading depth slowed but remained constant until the end of the fifth grade.

Overall, second-order PLGC models can be very useful in the area of educational research where most often the interest of researchers is centered on student academic progress or changes in attitude and affect. Second-order PLGC models enable researchers to summarize individual behavior that exhibits distinct phases of development in each segment and thereby allow researchers to address key questions such as developmental studies or studies seeking to measure the effect of treatment/ intervention, and so forth, such as when individuals may need to seek professional services at the timing when mental ability decreases.

Additionally, a second-order PLGC model can estimate the unknown location of the knot, which can enhance the ability of researchers to estimate when a treatment/intervention should be introduced so as to maximize its effects.

ACKNOWLEDGMENTS

The research reported here was funded by a grant from the Institute of Education Sciences, U.S. Department of Education, to the University of Maryland (R305A090152). The opinions expressed are those of the authors and do not represent views of the institute or the U.S. Department of Education.

REFERENCES

Anglin, J. M. (1993). Vocabulary development: A morphological analysis. *Monographs of Society Research in Child Development, 58*(10, Serial No. 238).

August, D., Kenyon, D., Malabonga, V., Louguit, M., & Caglarcan, S. (2001). *Extract the Base.* Washington, DC: Center for Applied Linguistics.

Bates, D. M., & Watts, D. G. (1988). *Nonlinear regression analysis and its applications*. New York, NY: Wiley.

Blozis, S. A. (2006). A second-order structured latent curve model for longitudinal data. In K. van Montfort, H. Oud, & A. Satorra (Eds.), *Longitudinal models in the behavioral and related sciences* (pp. 189–214). Mahwah, NJ: Erlbaum.

Bollen, K. A., & Curran, P. J. (2006). *Latent growth models: A structural equation perspective*. Hoboken, NJ: Wiley.

Carlisle, J. F. (1988). Knowledge of derivational morphology and spelling ability in fourth, sixth, and eighth graders. *Applied Psycholinguistics, 9,* 247–266.

Choi, J., Harring, J. R., & Hancock, G. R. (2009). Latent growth modeling for logistic response functions. *Multivariate Behavioral Research, 44,* 620–645.

Chou, C.-P., Yang, D., Pentz, M. A., & Hser, Y.-I. (2004). Piecewise growth curve modeling approach for longitudinal prevention study. *Computational Statistics & Data Analysis, 46,* 213–225.

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Erlbaum.

Cudeck, R. (1996). Mixed-effects models in the study of individual differences with repeated measures data. *Multivariate Behavioral Research, 31,* 371–403.

Cudeck, R., & Codd, C. L. (2012). A template for describing individual differences in longitudinal data with application to the connection between learning and ability. In J. R. Harring & G. R. Hancock (Eds.), *Advances in longitudinal methods in the social and behavioral sciences* (pp. 3–24). Charlotte, NC: Information Age.

Cudeck, R., & Harring, J. R. (2010). Developing a random coefficient model for nonlinear repeated measures data. In S.-M. Chow, E. Ferrer, & F. Hsieh (Eds.), *Statistical methods for modeling human dynamics: An interdisciplinary dialogue* (pp. 289–318). New York, NY: Routledge.

Cudeck, R., & Klebe, K. J. (2002). Multiphase mixed-effects models for repeated measures data. *Psychological Methods, 7,* 41–63.

Dimitruk, P., Schermelleh-Engel, K., Kelava, A., & Moosbrugger, H. (2007). Challenges in nonlinear structural equation modeling. *Methodology, 3,* 100–114.

Duncan, T. E., Duncan, S. C., & Strycker, L. A. (2006). *An introduction to latent variable growth curve modeling: Concepts, issues, and applications* (2nd ed.). Mahwah, NJ: Erlbaum.

Ferrer, E., Balluerka, N., & Widaman, K. F. (2008). Factorial invariance and the specification of second-order latent growth models. *Methodology, 4,* 22–36.

Fitzmaurice, G. M., Laird, N. M., and Ware, J. H. (2011). *Applied longitudinal analysis* (2nd ed.). New York, NY: Wiley.

Grimm, K. J., & Ram, N. (2009). Nonlinear growth models in Mplus and SAS. *Structural Equation Modeling: A Multidisciplinary Journal, 16,* 676–701.

Hancock, G. R., & Buehl, M. M. (2008). Second-order latent growth models with shifting indicators. *Journal of Modern Applied Statistical Methods, 7,* 39–55.

Hancock, G. R., Kuo, W.-L., & Lawrence, F. R. (2001). An illustration of second-order latent growth models. *Structural Equation Modeling: A Multidisciplinary Journal, 8,* 470–483.

Harring, J. R., Cudeck, R., & du Toit, S. H. C. (2006). Fitting partially nonlinear random coefficient models as SEMs. *Multivariate Behavioral Research, 41,* 579–596.

Harring, J. R., Kohli, N., Silverman, R. D., & Speece, D. L. (2012). A second-order conditionally linear mixed effects model with observed and latent variable covariates. *Structural Equation Modeling: A Multidisciplinary Journal, 19,* 118–136.

Hu, L. & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling, 6,* 1–55.

Kohli, N. (2011). *Estimating unknown knots in piecewise linear-linear latent growth mixture models* (Unpublished doctoral dissertation). University of Maryland, College Park.

Lawley, D. N., & Maxwell, A. E. (1971). *Factor analysis as a statistical method* (2nd ed.). London, UK: Butterworths.

Marsh, L. C., & Cormier, D. R. (2001). *Spline regression models.* Thousand Oaks, CA: Sage.

Meredith, W., & Tisak, J. (1990). Latent curve analysis. *Psychometrika, 55,* 107–122.

Muthén, L. K., & Muthén, B. O. (1998–2010). *Mplus user's guide* (6th ed.). Los Angeles, CA: Author.

Oehlert, G. W. (1992). A note on the delta method. *The American Statistician, 46,* 27–29.

Pierce, C. A., Block, R. A., & Aguinis, H. (2004). Cautionary note on reporting eta-squared values from multifactor ANOVA designs. *Educational and Psychological Measurement, 64,* 916–924.

Preacher, K. J., & Hancock, G. R. (2012). On interpretable reparameterizations of linear and nonlinear latent growth curve models. In J. R. Harring & G. R. Hancock (Eds.), *Advances in longitudinal methods in the social and behavioral sciences* (pp. 25–58). Charlotte, NC: Information Age.

Preacher, K. J., Wichman, A. L., MacCallum, R. C., & Briggs, N. E. (2008). *Latent growth curve modeling*. Thousand Oaks, CA: Sage.

Proctor, C. P., Silverman, R. D., Harring, J. R., & Montecillo, C. (2011). The role of vocabulary depth in predicting reading comprehension among English monolingual and Spanish-English bilingual children in elementary school. *Reading and Writing: An Interdisciplinary Journal, 25,* 1635–1664. doi: 10.1007/s11145-011-9336-5

R Development Core Team (2009). R: A language and environment for statistical computing. R Foundation for Statistical Computing. Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org

Sapp, M. (2006). *Basic psychological measurement, research designs, and statistics without math.* Springfield, IL: Charles C. Thomas.

Sayer, A. G., & Cumsille, P. E. (2001). Second-order latent growth models. In L. M. Collins & A. G. Sayer (Eds.), *New methods for the analysis of change* (pp. 179–200). Washington, DC: American Psychological Association.

Seber, G. A. F., & Wild, C. J. (1989). *Nonlinear regression*. New York, NY: Wiley.

Semel, E. M., Wiig, E. H., & Secord, W. (2003). *Clinical evaluation of language fundamentals.* San Antonio, TX: Psychological Corporation.

Verbeke, G., & Molenberghs, G. (2000). *Linear mixed models for longitudinal data*. New York, NY: Springer.

Weiss, B. A. (2010). *A comparison of methods for testing for interaction effects in structural equation modeling* (Unpublished doctoral dissertation). University of Maryland, College Park.

APPENDIX A

The Procedure of Reparameterization

When the mean of the slope of the first phase is greater than the mean of slope of the second phase, the reparameterized model can be written as the minimum of the two segments. That is,

$$g = \min(l_1(t), l_2(t)).$$

Harring, Cudeck, and du Toit (2006) demonstrated that the *min* function could be conveniently written as follows:

$$\min(l_1, l_2) = \frac{1}{2}(l_1 + l_2 - \sqrt{(l_1 - l_2)^2}).$$

Substituting the segments from Equation 6 into the aforementioned expression gives the following:

$$\begin{aligned} g &= \min(l_1(t), l_2(t)) \\ &= \frac{1}{2}(l_1 + l_2 - \sqrt{(l_1 - l_2)^2}) \\ &= \frac{1}{2}(\alpha_{i1} + \alpha_{i2}t_j + \alpha_{i3} + \alpha_{i4}t_j - \sqrt{(\alpha_{i1} + \alpha_{i2}t_j - (\alpha_{i3} + \alpha_{i4}t_j))^2}) \\ &= \frac{1}{2}(\alpha_{i1} + \alpha_{i3} + (\alpha_{i2} + \alpha_{i4})t_j - \sqrt{(\alpha_{i1} - \alpha_{i3} + (\alpha_{i2} + \alpha_{i4})t_j))^2}). \end{aligned}$$

When $t_j = \gamma$, then $\alpha_{i1} + \alpha_{i2}\gamma = \alpha_{i3} + \alpha_{i4}\gamma$. Thus, $\alpha_{i1} - \alpha_{i3} = (\alpha_{i4} - \alpha_{i2})\gamma$. Through substitution, we see that

$$\begin{aligned} g &= \min(l_1(t), l_2(t)) \\ &= \frac{1}{2}(\alpha_{i1} + \alpha_{i3} + (\alpha_{i2} + \alpha_{i4})t_j - \sqrt{((\alpha_{i4} - \alpha_{i2})\gamma + (\alpha_{i2} - \alpha_{i4})t_j))^2}) \\ &= \frac{1}{2}(\alpha_{i1} + \alpha_{i3} + (\alpha_{i2} + \alpha_{i4})t_j - \sqrt{((\alpha_{i2} - \alpha_{i4})(t_j - \gamma))^2}) \\ &= \frac{1}{2}(\alpha_{i1} + \alpha_{i3} + (\alpha_{i2} + \alpha_{i4})t_j + (\alpha_{i4} - \alpha_{i2})\sqrt{(t_j - \gamma)^2}). \end{aligned}$$

Therefore, $\beta_{i1} = (\alpha_{i1} + \alpha_{i3})/2$, $\beta_{i2} = (\alpha_{i2} + \alpha_{i4})/2$, and $\beta_{i3} = (\alpha_{i4} - \alpha_{i2})/2$, and the reparameterization is like that in Equation 7. Unlike the functional form in Equation 6, it is the function form in Equation 7 that will be fitted using standard SEM software.

APPENDIX B

The Multivariate Delta Method

The multivariate delta method transforms the estimated variances of β_1, β_2, and β_3 (i.e., $\hat{\sigma}^2_{\beta_1}$, $\hat{\sigma}^2_{\beta_2}$, and $\hat{\sigma}^2_{\beta_3}$) back to the respective variances of α_1, α_2, and α_4 (i.e., $\hat{\sigma}^2_{\alpha_1}$, $\hat{\sigma}^2_{\alpha_2}$, and $\hat{\sigma}^2_{\alpha_4}$) in the following way:

$$\hat{\alpha}_1 = \hat{\beta}_1 + \hat{\beta}_3\hat{\gamma} = g_1$$

$$\hat{\alpha}_2 = \hat{\beta}_2 - \hat{\beta}_3 = g_2$$

$$\hat{\alpha}_4 = \hat{\beta}_2 + \hat{\beta}_3 = g_3.$$

Thus, $\hat{\alpha}_1 = \hat{\beta}_1 + \hat{\beta}_3\hat{\gamma} = 27.73 - 2.05(1.41) = 24.84$.

The variance of the back-transformed parameter, $\hat{\alpha}_1$, can be computed as $Var(\hat{\sigma}^2_{\alpha_1}) = \mathbf{d}'\boldsymbol{\Sigma}\mathbf{d}$, where $\mathbf{d}$ is a vector of partial derivatives of g_1 with respect to β_1, β_2, and β_3, respectively, and $\boldsymbol{\Sigma}$ is the matrix of variance and covariance terms of $\boldsymbol{\beta}$.

$$Var(\hat{\sigma}^2_{\alpha_1}) = \mathbf{d}'\boldsymbol{\Sigma}\mathbf{d}$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial \beta_1} & \frac{\partial g_1}{\partial \beta_2} & \frac{\partial g_1}{\partial \beta_3} \end{bmatrix} \cdot \begin{bmatrix} \hat{\sigma}^2_{\beta_1} & & \\ \hat{\sigma}_{\beta_2,\beta_1} & \hat{\sigma}^2_{\beta_2} & \\ \hat{\sigma}_{\beta_3,\beta_1} & \hat{\sigma}_{\beta_3,\beta_2} & \hat{\sigma}^2_{\beta_3} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial g_1}{\partial \beta_1} \\ \frac{\partial g_1}{\partial \beta_2} \\ \frac{\partial g_1}{\partial \beta_3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ \hat{\gamma} \end{bmatrix} \cdot \begin{bmatrix} \hat{\sigma}^2_{\beta_1} & & \\ \hat{\sigma}_{\beta_2,\beta_1} & \hat{\sigma}^2_{\beta_2} & \\ \hat{\sigma}_{\beta_3,\beta_1} & \hat{\sigma}_{\beta_3,\beta_2} & \hat{\sigma}^2_{\beta_3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \hat{\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1.41 \end{bmatrix} \cdot \begin{bmatrix} 0.524 & & \\ -0.220 & 0.221 & \\ -0.060 & -0.164 & 0.238 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1.41 \end{bmatrix} = 1.166.$$

Thus, the standard error of α_1 is $SE(\alpha_1) \approx \sqrt{1.166} = 1.08$. Standard errors of the other transformed regression coefficients, α_2, and α_4, could be computed in a similar fashion. It should be noted that $\mathbf{\Sigma}$ can be obtained in Mplus by using the TECH3 option in the output statement.

APPENDIX C

Annotated M*plus* Input for Generic Second-Order Latent Piecewise Model

```
TITLE: 2nd-order piecewise model
DATA:  FILE IS data.dat;
VARIABLE: NAMES ARE y1-y27;
ANALYSIS: ESTIMATOR = ML;
     ITERATIONS = 10000;
     SDITERATIONS = 500;
     H1ITERATIONS = 10000;
     CONVERGENCE = .001;
     H1CONVERGENCE = .001;
MODEL:
!Measurement Portion of the PLGM
t1 BY
   y1
   y2*.7(1)
   y3*.7(2);
t2 BY
   y4
   y5*.7(1)
   y6*.7(2);
t3 BY
   y7
   y8*.7(1)
   y9*.7(2);
t4 BY
   y10
   y11*.7(1)
   y12*.7(2);
t5 BY
   y13
   y14*.7(1)
   y15*.7(2);
t6 BY
   y16
```

```
   y17*.7(1)
   y18*.7(2);
t7 BY
   y19
   y20*.7(1)
   y21*.7(2);
t8 BY
   y22
   y23*.7(1)
   y24*.7(2);
t9 BY
   y25
   y26*.7(1)
   y27*.7(2);

y1-y27*;
[y1@0 y4@0 y7@0 y10@0 y13@0 y16@0 y19@0 y22@0 y25@0];
[y2 y5 y8 y11 y14 y17 y20 y23 y26](2);
[y3 y6 y9 y12 y15 y18 y21 y24 y27](3);

!Structural Portion of the PLGM
w1 BY t1-t9@1;
w2 BY t1@0 t2@1 t3@2 t4@3 t5@4 t6@5 t7@6 t8@7 t9@8;
w3 BY t1* (p1); !Column 3 of design matrix
w3 BY t2-t9* (p2-p9);
w1*10(v1);
w2*1(v2);
w3*.5(v3);
w1 WITH w2*0;
w1 WITH w3*0;
w2 WITH w3*0;
[w1* w2* w3*];
[t1-t9@0];
t1-t9*(vard);

MODEL CONSTRAINT:
NEW (gam*);
v1 > 0;
v2 > 0;
v3 > 0;
vard > 0;
```

```
p1 = (sqrt((gam)^2));
p2 = (sqrt((1-gam)^2));
p3 = (sqrt((2-gam)^2));
p4 = (sqrt((3-gam)^2));
p5 = (sqrt((4-gam)^2));
p6 = (sqrt((5-gam)^2));
p7 = (sqrt((6-gam)^2));
p8 = (sqrt((7-gam)^2));
p9 = (sqrt((8-gam)^2));

OUTPUT: SAMPSTAT;
```