Multivariate Behavioral Research

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/hmbr20

Modeling Nonlinear Change via Latent Change and Latent Acceleration Frameworks: Examining Velocity and Acceleration of Growth Trajectories

Kevin Grimm [a], Zhiyong Zhang [b], Fumiaki Hamagami [c] & Michèle Mazzocco [d]

[a] Department of Psychology, University of California, Davis

[b] Department of Psychology, Notre Dame University

[c] Department of Psychiatry, University of Hawaii at Manoa

[d] Institute of Child Development, University of Minnesota

Version of record first published: 29 Mar 2013.

Multivariate Behavioral Research, 48:117–143, 2013

ISSN: 0027-3171 print/1532-7906 online
DOI: 10.1080/00273171.2012.755111

Modeling Nonlinear Change via Latent Change and Latent Acceleration Frameworks: Examining Velocity and Acceleration of Growth Trajectories

Kevin Grimm
Department of Psychology
University of California, Davis

Zhiyong Zhang
Department of Psychology
Notre Dame University

Fumiaki Hamagami
Department of Psychiatry
University of Hawaii at Manoa

Michèle Mazzocco
Institute of Child Development
University of Minnesota

We propose the use of the latent change and latent acceleration frameworks for modeling nonlinear growth in structural equation models. Moving to these frameworks allows for the direct identification of *rates of change* and *acceleration* in latent growth curves—information available indirectly through traditional growth curve models when change patterns are nonlinear with respect to time. To illustrate this approach, exponential growth models in the three frameworks are fit to longitudinal response time data from the Math Skills Development Project (Mazzocco & Meyers, 2002, 2003). We highlight the additional information gained from fitting growth curves in these frameworks as well as limitations and extensions of these approaches.

Correspondence concerning this article should be addressed to Kevin Grimm, Department of Psychology, University of California, Davis, One Shields Avenue, Davis, CA 95616. E-mail: kjgrimm@ucdavis.edu

Growth modeling is an analytic approach for understanding within-person change and between-person differences in within-person change. The linear growth model, commonly fit because of its simplicity and interpretability, decomposes individual observed trajectories into an *intercept*, representing an individual's predicted performance (e.g., score) at a specific point in time (often the first observation), and a time-invariant *linear slope*, representing an individual's rate of change over the observation period. Each individual's rate of change or velocity is constant across time, but this constant rate of change is allowed to vary over individuals. Thus, the study of between-person differences in the *linear slope* conforms to studying between-person differences in a constant *rate of change* across all points in time, and studying between-person differences in the *rate of change* is a primary interest to researchers studying longitudinal data.

When modeling change that is nonlinear with respect to time, between-person differences in the rate of change are more difficult to study directly because the rate of change is not constant across time and the rate of change is often a combination of multiple latent variables (e.g., linear and quadratic slopes in a quadratic growth curve). When discussing growth models of nonlinear change, we represent the most basic to the most complex forms of nonlinearity. That is, we represent (a) models that are only nonlinear with respect to time but linear with respect to parameters and random coefficients, such as the quadratic ($y_{nt} = b_{0n} + b_{1n} \cdot t + b_{1n} \cdot t^2$), log-time ($y_{nt} = b_{0n} + b_{1n} \cdot \log(t)$), and latent basis ($y_{nt} = b_{0n} + b_{1n} \cdot \alpha_t(t)$) models, where one or more functions or transformations of time are included; (b) models that are nonlinear with respect to time and nonlinear with respect to parameters but linear with respect to random coefficients, such as the following exponential model: $y_{nt} = b_{0n} + b_{1n} \cdot \exp(\alpha \cdot t)$, where α is an estimated parameter; and (c) models that are nonlinear with respect to time, random coefficients, and/or parameters, such as the following exponential model: $y_{nt} = b_{0n} + b_{1n} \cdot \exp(b_{2n} \cdot t)$. We note that, regardless of model complexity, when there is a single between-person variable affecting time (e.g., $y_{nt} = b_{0n} + b_{1n} \cdot \log(t)$, $y_{nt} = b_{0n} + b_{1n} \cdot \exp(\alpha \cdot t)$, or $y_{nt} = b_{0n} + t \wedge (b_{1n})$), the variability in that single between-person variable (e.g., b_{1n}) perfectly reflects between-person differences in the rate of change even though the parameter itself does not represent the rate of change and the rate of change varies with time. However, any time there are two or more between-person variables affecting time (e.g., $y_{nt} = b_{0n} + b_{1n} \cdot t + b_{2n} \cdot t^2$ or $y_{nt} = b_{0n} + b_{1n} \cdot \exp(b_{2n} \cdot t)$, the rate of change is a complex combination of these random coefficients and it is these models where these issues are magnified.

The goals of this article are twofold. First, we introduce a method in which the rate of change and between-person differences in the rate of change can be studied directly in growth models that are nonlinear with respect to time using latent change score models. Second, we extend this method to study acceleration and between-person differences in acceleration. We continue with a discussion of growth models with nonlinear trajectories highlighting how the rate of change is not directly parameterized with an illustrative example involving the exponential growth model. We then describe the latent change score framework, how growth models with nonlinear trajectories can be fit in this framework, and how it can be used to study between-person differences in the rate of change. We then discuss the latent acceleration framework, how models can be estimated within this framework, and how it can be used to study between-person differences in the rate of change and acceleration. Finally, we illustrate these approaches using longitudinal data from the Math Skills Development Project (Mazzocco & Myers, 2002, 2003), a prospective longitudinal study of cognitive correlates of mathematics ability, to study between-person differences in the rate of change and acceleration in mathematics-related skills.

MODELING NONLINEAR CHANGE

Growth curve models with nonlinear trajectories have parameters that describe specific features of the nonlinear curve. Certain models, such as those based on the exponential, logistic, and Gompertz functions, have parameters that map onto theoretically meaningful aspects of the curve, such as its rate of change at a specific point in time, asymptotic level, and rate of approach to the asymptotic level. Other models, such as those based on power functions and polynomials, have parameters that are difficult to map onto theoretically meaningful aspects of curve (Cudeck & du Toit, 2002). Often researchers attempt to find a balance between having a model that fits the data well with a model that yields parameters that are interpretable and of substantive interest. A primary example of this issue comes from the modeling of human growth. Certain models (e.g., Preece & Baines, 1978) were developed to adequately account for the data and do so with parameters that map onto known between-person differences (e.g., timing of pubertal growth) whereas other models were developed with a greater focus on data-model fit (e.g., Jolicoeur, Pontier, Pernin, & Sempé, 1988).

To illustrate how nonlinear models, even those with interpretable and theoretically meaningful parameters, are unable to parameterize the *rate of change* we discuss an exponential model that is relatively simple and commonly used in applied research (e.g., Burchinal & Appelbaum, 1991). One version of the exponential model can be written as

$$\begin{aligned} Y_{nt} &= y_{nt} + u_{nt} \\ y_{nt} &= b_{0n} + b_{1n} \cdot (1 - \exp(-b_{2n} \cdot t)). \end{aligned} \tag{1}$$

The first part of Equation 1 decomposes the observed score for person n at time t (Y_{nt}) into its true (y_{nt}) and unique (u_{nt}) scores following core ideas from Classical Test Theory. The second part of Equation 1 is an exponential trajectory equation for the true scores. In the trajectory equation b_{0n} is the intercept or predicted score when $t = 0$ for individual n, b_{1n} is the total change from the intercept to the asymptotic level for individual n, and b_{2n} is the rate of approach to the asymptote for individual n. The parameters of the exponential model completely describe its shape and, at the same time, highlight certain features of the nonlinear curve—an initial score at $t = 0$ (b_{0n}), how much change is expected to occur because $t = 0$ (b_{1n}), and how quickly the asymptote is approached (b_{2n}). Additionally, certain parameters can be combined to understand other features of the curve. For example, the individual asymptotic level is equal to $b_{0n} + b_{1n}$ and the individual rate of change at $t = 0$ is $b_{1n} \cdot b_{2n}$. Finally, the exponential model can be reparameterized to highlight other aspects of the curve (see Preacher & Hancock, 2012) based on a researcher's substantive interests. For example, if the asymptotic level is of particular interest, then the following trajectory equation can be specified as

$$y_{nt} = b_{0n} + (b_{1n} - b_{0n}) \cdot (1 - \exp(-b_{2n} \cdot t)), \tag{2}$$

where b_{1n} is now the asymptotic level as opposed to the change from $t = 0$ to the asymptotic level.

The exponential model of Equation 1 is a nonlinear random coefficient model (fully nonlinear mixed model) and cannot be directly estimated within the structural equation modeling framework because b_{1n} and b_{2n} enter the model in a nonlinear fashion. The nonlinear random coefficient model of Equation 1 can, however, be *approximated* within the structural equation modeling framework by linearizing the target (mean) function through a first-order Taylor series expansion (see Beal & Sheiner, 1982; Browne, 1993; Browne & du Toit, 1991; Grimm, Ram, & Hamagami, 2011). Briefly, the model of Equation 1 can be reexpressed with latent variables mapping onto b_{0n}, b_{1n}, and b_{2n} with factor loadings equivalent to the partial derivatives of the target function with respect to each mean. The factor loadings of latent variables are complex nonlinear functions; however, they only vary with time making the model estimable using general structural equation modeling software.

Following Browne & du Toit (1991), the target function of the exponential model of Equation 1 is

$$\mu_y = \mu_0 + \mu_1 \cdot (1 - \exp(-\mu_2 \cdot t)). \tag{3}$$

Taking the partial derivative of Equation 3 with respect to each parameter, the exponential model at the individual level can be reexpressed as a linear combination of latent variables, such as

$$y_{nt} = x_{0n} + x_{1n} \cdot (1 - \exp(-\mu_2 \cdot t)) + x_{2n} \cdot (\mu_1 \cdot t \cdot \exp(-\mu_2 \cdot t)), \tag{4}$$

where x_{0n} represents the value of y_{nt} when $t = 0$, x_{1n} represents change from x_{0n} to the asymptotic level, $(1 - \exp(-\mu_2 \cdot t))$ is the partial derivative of the target function with respect to μ_1, μ_2 is the mean of the rate parameter b_{2n}, x_{2n} represents the rate of approach to the asymptotic level, $(\mu_1 \cdot t \cdot \exp(-\mu_2 \cdot t))$ is the partial derivative of the target function with respect to μ_2, and μ_1 is the mean of b_{1n} and the latent variable x_{1n}. The mean of x_{2n} is fixed to 0 because this latent variable affects only the covariance structure of the exponential model and does not alter its mean structure. Note that *x*s, and not *b*s, are contained in Equation 4 because there is not a one-to-one mapping of *x*s and *b*s. For example, the mean of b_{2n} is μ_2 and the mean of x_{2n} is 0. The model of Equation 4 can be estimated within the structural equation modeling framework because the latent variables (x_{0n}, x_{1n}, and x_{2n}) enter the model in a linear (additive) fashion.

A path diagram of the model of Equation 4 is contained in Figure 1. In this diagram, latent true scores are drawn separately from the observed scores to highlight how the trajectory is for the latent true scores. The intercept, x_0, has

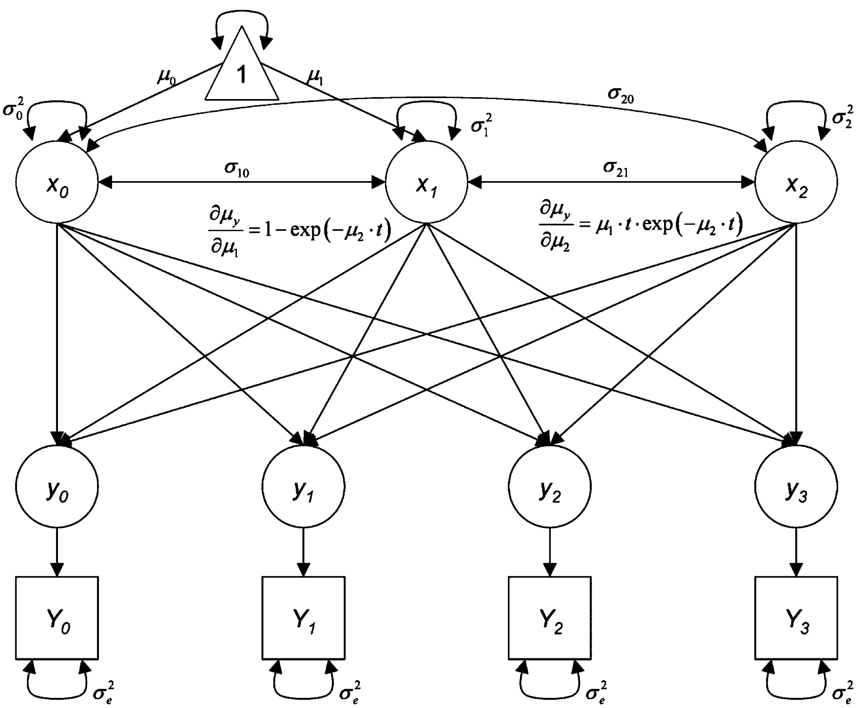


FIGURE 1 Path diagram of an exponential growth model in the traditional latent growth modeling framework.

factor loadings equal to 1; x_1 has factor loadings equal to $1 - \exp(-\mu_2 \cdot t)$; and x_2 has factor loadings equal to $\mu_1 \cdot t \cdot \exp(-\mu_2 \cdot t)$. The latent variables x_0 and x_1 have means (one-headed arrows from the triangle) and all latent variables have variances (two-headed arrows to and from the same variable) and covariances (two-headed arrow between latent variables). The mean of b_{2n} from Equation 1 (μ_2) is estimated through the factor loadings of x_1 and x_2. The factor loadings for x_1 and x_2 are set equal to their respective partial derivatives of the target function using nonlinear constraints, which are available in most structural equation modeling programs (see Grimm & Ram, 2009).

We note that growth curves with nonlinear trajectories, such as the exponential model described earlier, do not have parameters that directly reflect the *rate of change* because the rate of change is constantly changing with time. A key feature of many nonlinear models is that the rate of change is a combination of multiple latent variables (random coefficients within the mixed-effects framework). Thus, there is no single latent variable that maps onto the rate of change (as the linear slope does in the linear growth model) and therefore, there is not a latent variable that, by itself, captures the between-person differences in the rate of change. An additional example of this can be seen with the quadratic growth model, where between-person variations in the linear and quadratic slopes combine to create between-person differences in the rate of change. A limitation of this traditional approach to fitting nonlinear growth models is that the *rate of change* is not parameterized in the model even though researchers have a fundamental interest in the *rate of change*.

LATENT CHANGE SCORE MODELS

Latent change score modeling (LCS; McArdle, 2001, 2009; McArdle & Hamagami, 2001), sometimes referred to latent difference score modeling, is a framework for studying longitudinal change. Often LCS models are used to examine time-sequential dependency in multivariate longitudinal data because these models combine the time-sequential features of autoregressive cross-lag (e.g., Jöreskog, 1970, 1974) models and the change modeling features of latent growth models (e.g., McArdle & Epstein, 1987; Meredith & Tisak, 1990) for longitudinal panel data. In this section, we describe how the latent change score framework can be used to study between-person differences in the *rate of change* in nonlinear models.

In LCS models, as with traditional growth curve models, observed scores at time t are decomposed into theoretical true scores and unique scores written as

$$Y_{nt} = y_{nt} + u_{nt}. \tag{5}$$

Instead of a trajectory equation for the latent true scores (as in Equation 1), the latent true scores have an autoregressive relationship such that the true score at time t is equal to the true score at time $t-1$ plus the *change* that has occurred between the two. This can be written as

$$y_{nt} = y_{nt-1} + \Delta y_{nt}, \tag{6}$$

where y_{nt-1} is the latent true score at time $t-1$ and Δy_{nt} is the latent change score.

A trajectory equation is then written for the latent change scores (Δy_{nt}) as opposed to the latent true scores (y_{nt}). When specifying models for latent change scores, the derivative of the latent growth model with respect to time is needed because the latent change scores are the discrete analog of the first derivative of y_{nt}. Thus, to fit the exponential growth model in Equation 1 using the latent change score approach, we write

$$\Delta y_{nt} = b_{1n} \cdot b_{2n} \cdot \exp(-b_{2n} \cdot t), \tag{7}$$

where b_{2n} is the rate of approach to the asymptote as defined in Equation 1 and b_{1n} is a rotated version of the change from $t = 0$ to the upper asymptote (see Zhang, McArdle, & Nesselroade, 2012).

The model of Equations 5–7 can be fit within the structural equation modeling framework through the same process of linearization with Taylor Series Expansion. The model for latent difference scores of Equation 7 can be expanded as

$$\Delta y_{nt} = x_{1n} \cdot (\mu_2 \cdot \exp(-\mu_2 \cdot t)) + x_{2n} \cdot (-(\mu_2 \cdot \mu_1 \cdot t - \mu_1) \cdot \exp(-\mu_2 \cdot t)). \tag{8}$$

The factor loadings for x_{1n} and x_{2n} are partial derivatives of the first derivative of the target function as well as the derivatives of the factor loadings in Equation 4 with respect to t. A path diagram of the exponential model based on latent change scores is contained in Figure 2. In this diagram, the latent true scores ($y_0 - y_3$) have a fixed unit autoregressive relationship to create the latent change scores ($\Delta y_1 - \Delta y_3$). The latent intercept, x_0, feeds into the first latent true score and x_1 and x_2 are indicated by the latent change scores with factor loadings described in Equation 8. Thus, the factor loadings for x_1 equal $\mu_2 \cdot \exp(-\mu_2 \cdot t)$ and the factor loadings for x_2 equal $-(\mu_1 \cdot \mu_2 \cdot t - \mu_1) \cdot \exp(-\mu_2 \cdot t)$. As in Figure 1, x_0 and x_1 have means and x_0, x_1, and x_2 have variances and covariances. Fitting the exponential model of Equation 1 using the traditional growth modeling approach or Equation 7 using the latent change score approach will result in the same model expectations and fit. That is, this way of approaching the estimation of the exponential model does not change the model-implied trajectory or individual differences in the trajectory. However, the latent change

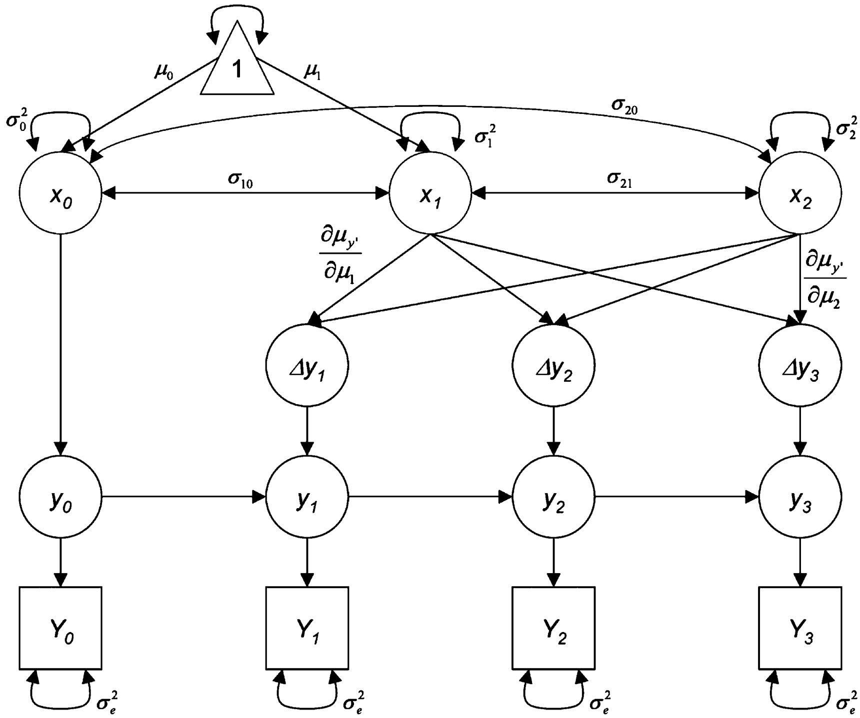


FIGURE 2 Path diagram of an exponential growth model in the latent change score framework.

approach provides direct information regarding the *rate of change* because the latent change scores are the outcome and represent the *rate of change* at each measurement occasion, instead of the latent true scores, which are the outcome with traditionally specified growth models and represent *status* or *position* at each measurement occasion. Sample level information regarding the rate of change can be found by calculating mean and variance expectations of the latent change scores.

LATENT ACCELERATION SCORE MODELS

Latent acceleration score (LAS; Hamagami & McArdle, 2007) models take the LCS models one step further by examining changes in the rate of change or acceleration, that is,

$$\Delta y_{nt} = \Delta y_{nt-1} + \Delta\Delta y_{nt}, \tag{9}$$

where Δy_{nt} is the latent change score at time t, Δy_{nt-1} is the latent change score at time $t-1$, and $\Delta\Delta y_{nt}$ is the change in the rate of change between adjacent times $t-1$ and t or the *acceleration*. Specifying growth models in the LAS framework involves specifying a trajectory equation for the latent acceleration scores ($\Delta\Delta y_{nt}$). The trajectory equation for the latent acceleration scores is specified using the second derivative of the latent growth model with respect to time. Thus, to fit the exponential growth model in Equation 1 using the latent acceleration approach, we specify

$$\Delta\Delta y_{nt} = b_{1n} \cdot (-b_{2n}^2) \cdot \exp(-b_{2n} \cdot t), \quad (10)$$

where b_{2n} is the individual rate of approach to the asymptote as defined in Equation 1 and b_{1n} is a rotated version of the change from $t = 0$ to the upper asymptote.

The model of Equations 9 and 10 can be fit within the structural equation modeling framework through the same process of linearization with Taylor Series Expansion. The model for latent acceleration scores of Equation 10 can be expanded as

$$\Delta\Delta y_{nt} = x_{1n}\cdot(-\mu_2^2\cdot\exp(-\mu_2\cdot t))+x_{2n}\cdot((\mu_2^2\cdot\mu_1\cdot t-2\cdot\mu_2\cdot\mu_1)\cdot\exp(-\mu_2\cdot t)). \quad (11)$$

The factor loadings for x_{1n} and x_{2n} are partial derivatives of the second derivative of the target function as well as the second derivatives of the factor loadings in Equation 4 with respect to t. A path diagram of the exponential model based on latent acceleration scores (Equation 11) is contained in Figure 3. The latent true scores ($y_0 - y_3$) have a fixed unit autoregressive relationship to create the latent change scores ($\Delta y_1 - \Delta y_3$), which also have a fixed unit autoregressive relationship to create the latent acceleration scores ($\Delta\Delta y_2 - \Delta\Delta y_3$). The latent intercept, x_0, feeds into the first latent true score and x_1 and x_2 are indicated by the latent acceleration scores with factor loadings equal to $-\mu_2^2 \cdot \exp(-\mu_2 \cdot t)$ and $(\mu_2^2 \cdot \mu_1 \cdot t - 2 \cdot \mu_2 \cdot \mu_1) \cdot \exp(-\mu_2 \cdot t)$, respectively. Additionally, x_1 and x_2 are indicated by the latent change score at Time 2. The constraints for these factor loadings can be determined by comparing model expectations for the LAS and LCS models.[1] For the exponential model, the factor loadings for x_1 and x_2 equal $\frac{\mu_2^2\cdot\exp(-\mu_2\cdot 2)}{1-\exp(-\mu_2)}$ and $-(\frac{\mu_1\cdot\mu_2(2\cdot\mu_2\cdot\exp(\mu_2)-2\cdot\exp(\mu_2)+2-\mu_2)}{\exp(\mu_2)\cdot(\exp(\mu_2)-1)^2})$, respectively. As in the previous models, x_0 and x_1 have means and x_0, x_1, and x_2 have variances and covariances.

Fitting the exponential growth model defined in Equation 1, Equation 7 using the latent change score approach, or Equation 10 using the latent acceleration

[1]This material can be found on the first author's website as well as information on how commonly fit growth models (e.g., linear, quadratic, cubic, latent basis) can be fit in the latent change and latent acceleration frameworks.

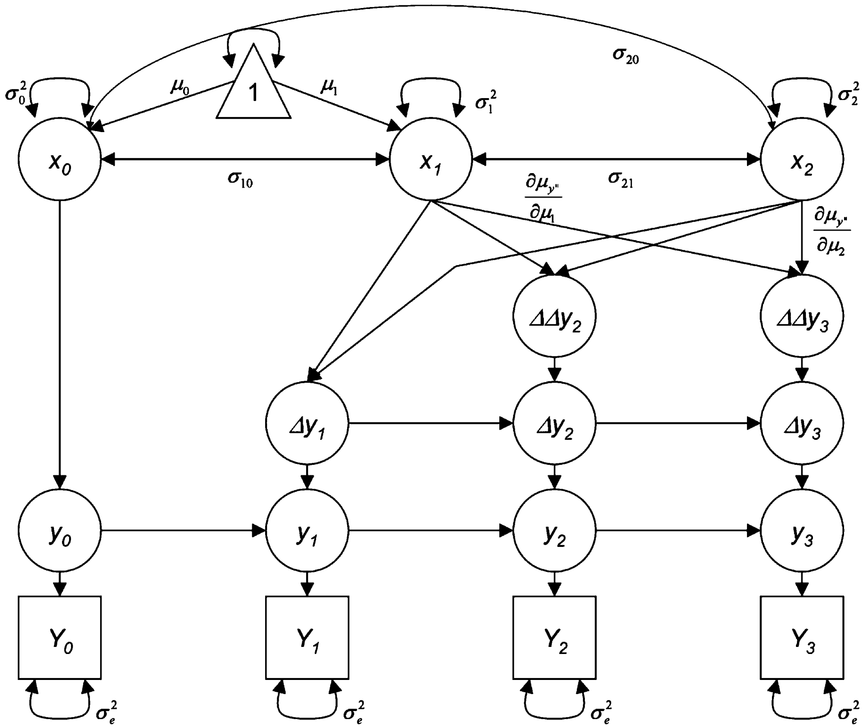


FIGURE 3 Path diagram of an exponential growth model in the latent acceleration score framework.

approach will result in the same model expectations and fit. However, a benefit of moving to the latent acceleration approach is the availability of direct information regarding both the *rate of change* and the *acceleration*, which can be directly examined and evaluated.

Covariates

The benefits of the latent change and latent acceleration approaches to modeling and understanding change are amplified when time-invariant covariates are added as predictors of b_{0n}, b_{1n}, and b_{2n} (similarly for x_{0n}, x_{1n}, and x_{2n}). In traditional growth curve modeling, the focus is often placed on whether a time-invariant covariate is predictive of certain aspects of the growth model (e.g., intercept, change to upper asymptote, rate of approach). For example, and building from the exponential growth model from Equation 1 (similar equations can be written for the exponential model based on latent changes and latent acceleration scores; Equation 7 and 10, respectively), if a time-invariant covariate Z_n (coded 0/1 for simplicity) is included as a predictor of b_{0n}, b_{1n}, and b_{2n}, we can write the following equations:

$$\begin{aligned} b_{0n} &= \gamma_{00} + \gamma_{10} \cdot Z_n + d_{0n} \\ b_{1n} &= \gamma_{01} + \gamma_{11} \cdot Z_n + d_{1n} \\ b_{2n} &= \gamma_{02} + \gamma_{12} \cdot Z_n + d_{2n}. \end{aligned} \tag{10}$$

In these equations, γ_{00} is the predicted score at time $t = 0$ when $Z_n = 0$, γ_{10} is the difference in the predicted score at time $t = 0$ for $Z_n = 1$, d_{0n} is the individual residual deviation for b_{0n}; γ_{01} is the predicted total change to the upper asymptote when $Z_n = 0$, γ_{11} is the difference in the total change to the upper asymptote for $Z_n = 1$, and d_{1n} is the individual residual deviation for b_{1n}; and γ_{02} is the predicted rate of approach to the asymptote when $Z_n = 0$, γ_{12} is the difference in the rate for $Z_n = 1$, and d_{2n} is the individual residual deviation for b_{2n}. Often the significance of γ_{10}, γ_{11}, and γ_{12} is the primary interest because these coefficients represent differences in specific aspects of the curve (e.g., intercept, change to the upper asymptote, and rate of approach to the asymptote) for the two groups.

An alternative approach to studying the effect of Z_n on the growth trajectory is through indirect effects. In traditional growth modeling, the indirect effect from Z_n to the true scores (y_{nt}) indicates the difference in *status* between the two groups at each measurement occasion. That is, the variable Z_n could be included in Figure 1 with directional effects to x_{0n}, x_{1n}, and x_{2n}. The indirect from Z_n to y_0, y_1, y_2, and y_3 go through x_{0n}, x_{1n}, and x_{2n} and these indirect effects indicate the degree to which the covariate, Z_n, affects status of the true score at each measurement occasion. In the latent change framework, these indirect effects can be examined as well as the indirect effects from Z_n to the latent change scores (Δy_{nt}), which would provide information on how the covariate is associated with changes in the *rate of change* of the trajectory over time. In the latent acceleration framework, the indirect effects from Z_n to the latent acceleration scores ($\Delta\Delta y_{nt}$) can also be studied providing information on how the covariate is associated with changes in the acceleration of the trajectory over time. These associations are often of interests to researchers but overlooked because the effects of Z_n on the *status*, *rate of change*, and *acceleration* of the growth trajectory vary across time in growth models that are nonlinear with respect to time. Additionally, the significance of these associations can vary over time.

ILLUSTRATIVE EXAMPLE

Data

Illustrative data come from the Math Skills Development Project (Mazzocco & Myers, 2002, 2003), a longitudinal study of cognitive correlates of mathematics achievement. Participants were recruited from 1 of 7 participating schools within the Baltimore County Public School Districts, a diverse district that included 100 elementary schools in the initial year of the study. Seven schools that were targeted for participation had a low mobility index (to decrease attrition) and a low rate of free/reduced lunch eligibility (used to screen for poverty) relative to the district average.

In addition, schools were selected to represent different geographical regions of this district. The participating schools represented a heterogeneous sample of neighborhoods, excluding those with the very highest and lowest levels of socioeconomic status. This deliberate omission served to reduce the presence of known influences on mathematics outcomes tied to socioeconomic status given the study's focus on cognitive correlates of mathematics achievement. In all, 57% of eligible participants enrolled (249 students; 120 boys), of which 88% were White and 7% were African American (Mazzocco & Myers, 2002).

Measures

Rapid Automatized Naming. The Rapid Automatized Naming task (RAN; Denckla & Rudel, 1974) is a lexical retrieval task that was administered each year from kindergarten through eighth grade. The RAN includes three subtests: Colors, Letters, and Numbers. For each subtest, a brief practice trial was followed by a timed test trial. A total of 50 stimuli were presented on one page, and the child was asked to name the stimuli (colored squares, letters, or one-digit numbers) as quickly as possible without error. The examiner recorded response times with a handheld stopwatch. Total response time (RT) for the Numbers subtest was used as the dependent variable in this illustration. A plot of the RT scores for the RAN Numbers subtest is contained in Figure 4. The RT scores show sharp decreases indicating the anticipated improvement in performance in early elementary school. Improvement slows during late primary school and through middle school as individuals approach their optimal level of performance.

Mathematical learning disability. Children were categorized into two groups based on whether they met criteria for mathematical learning disability (MLD) reported by Mazzocco and colleagues (Mazzocco & Myers, 2003; Murphy, Mazzocco, Hanich, & Early, 2007). Specifically, children were classified as having a deficient mathematical ability if they consistently scored at or below

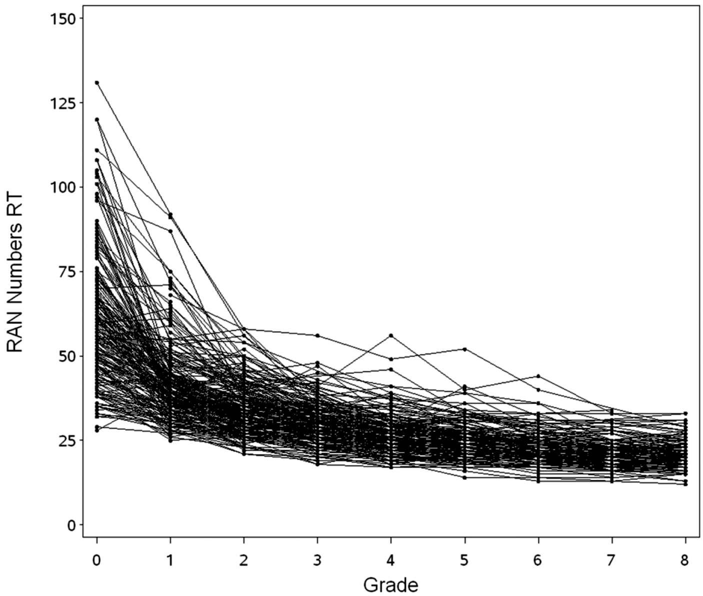


FIGURE 4 Longitudinal Rapid Automatized Naming Numbers response time scores from the Math Skills Development Project.

the 10th percentile (or within the 95th percentile confidence interval) relative to the study sample on either the Test of Early Mathematics–Third Edition or Woodcock-Johnson Revised Calculations subtest for the majority of the years during which they participated in the study. A total of 25 participants met these criteria, consistent with prevalence reports of MLD at ~6–10% of school-age children (e.g., Shalev, 2007). The remaining 224 children were classified as not having MLD.

Analytic Techniques

Various growth models were fit to the longitudinal RAN Numbers RT data using the approaches described earlier. Given the longitudinal trajectories several models may be appropriate including the latent basis (shape, free curve, unstructured) growth model, a power model ($y_{nt} = b_{0n} + b_{1n} \cdot t^{b_{2n}}$), and an exponential model. We present results from the exponential growth model discussed because of its common use, it is an inherently nonlinear model with two coefficients (latent variables) that affect the rate of change, and model fits were comparable. Thus, the traditional exponential growth model (Equation 1), the exponential growth model based on latent change scores (Equation 7), and the exponential growth model based on latent acceleration scores (Equation 10) were fit to the longitudinal RAN Numbers data in the structural equation modeling framework by linearization through Taylor Series Expansion (Equations 4, 8, and 11). One modification was made: the residual variance was forced to follow an exponential trend following notions from Browne and du Toit (1991; see also Grimm & Widaman, 2010). That is, the residual variance was expected to show sharp declines as response time performance improved and stabilized and the between-person differences in reaction time decreased. Forcing the residual variance to be equal across time led to a significant decrease in model fit; allowing the residual variance to be separately estimated at each grade led to a good fitting model but was less parsimonious. The goal of this first set of models was to illustrate how model fit is identical and how additional information can be gained from using the latent change and acceleration approaches.

Next, the same three models were fit with MLD (dummy coded 0/1 and then mean centered) as a time-invariant covariate and predictor of x_{0n}, x_{1n}, and x_{2n}. The predictor variable was mean centered because this is required to have proper estimation of covariate effects when using the Taylor Series Approximation (Grimm et al., 2011). The goal of this series of models was to illustrate how indirect effects from the time-invariant covariate to the latent true scores, latent change scores, and latent acceleration scores can aid in understanding the effects of such variables on predicted scores, rates of change, and acceleration. Mplus (v. 6.11) was used for all analyses with the full information maximum likelihood estimator to account for data incompleteness. All programming scripts are available on the first author's website.

RESULTS

The results section is presented in two parts. First, results from fitting the exponential model using the traditional growth modeling framework, the latent change framework, and the latent acceleration framework are presented. In this first section, we examine model fit statistics and parameter estimates and discuss the additional information gained through the latent change and acceleration frameworks. In the second section, we describe the results of using the three different frameworks to understand individual differences in the change pattern for the RAN RTs between normative children and children identified as having an MLD. Again we highlight the additional information gained through the use of the latent change and acceleration frameworks.

Exponential Growth Models

Fit statistics and parameter estimates from the exponential model fit in the three different frameworks are contained in Table 1. The fit of the exponential

TABLE 1
Parameter Estimates and Fit Statistics for Traditional Exponential Growth Model, Exponential Growth Model Based on Latent Change Scores, and Exponential Growth Model Based on Latent Acceleration Scores

Parameter	*Traditional Exponential*	*Latent Change Exponential*	*Latent Acceleration Exponential*
Mean estimates			
μ_0	56.395 (1.172)	56.395 (1.172)	56.395 (1.172)
μ_1	−36.696 (1.010)	−45.839 (1.504)	−57.261 (2.251)
μ_2	.430 (.016)	.430 (.016)	.430 (.016)
Factor loadings for x_{1n}			
$y1$	= .000	—	—
$y2, \Delta y2, \Delta y2$	.349 (.011)	.280 (.006)	.224 (.003)
$y3, \Delta y3, \Delta\Delta y3$	.576 (.014)	.182 (.001)	−.078 (.003)
$y4, \Delta y4, \Delta\Delta y4$	.724 (.013)	.118 (.001)	−.051 (.001)
$y5, \Delta y5, \Delta\Delta y5$	.821 (.012)	.077 (.002)	−.033 (.000)
$y6, \Delta y6, \Delta\Delta y6$	.883 (.009)	.050 (.002)	−.022 (.000)
$y7, \Delta y7, \Delta\Delta y7$	.924 (.007)	.033 (.002)	−.014 (.000)
$y8, \Delta y8, \Delta\Delta y8$	.951 (.006)	.021 (.002)	−.009 (.000)
$y9, \Delta y9, \Delta\Delta y9$	.968 (.004)	.014 (.001)	−.006 (.000)
Factor loadings for x_{2n}			
$y1$	= .000		
$y2, \Delta y2, \Delta y2$	−23.881 (.566)	−17.016 (.545)	−10.151 (.730)
$y3, \Delta y3, \Delta\Delta y3$	−31.082 (.926)	−2.734 (.656)	11.885 (.298)
$y4, \Delta y4, \Delta\Delta y4$	−30.342 (1.264)	3.648 (.526)	4.822 (.256)
$y5, \Delta y5, \Delta\Delta y5$	−26.327 (1.467)	5.906 (.317)	1.243 (.281)
$y6, \Delta y6, \Delta\Delta y6$	−21.417 (1.514)	6.142 (.164)	−.425 (.227)
$y7, \Delta y7, \Delta\Delta y7$	−16.725 (1.440)	5.493 (.147)	−1.079 (.153)
$y8, \Delta y8, \Delta\Delta y8$	−12.698 (1.291)	4.548 (.188)	−1.225 (.088)
$y9, \Delta y9, \Delta\Delta y9$	−9.444 (1.109)	3.593 (.210)	−1.137 (.043)
Variance/Covariance estimates			
σ_0^2	192.073 (29.365)	192.073 (29.365)	192.073 (29.365)
σ_1^2	138.217 (22.336)	268.945 (49.917)	538.809 (112.863)
σ_2^2	.019 (.005)	.019 (.005)	.019 (.005)
σ_{10}	−158.513 (25.185)	−218.507 (37.600)	−298.557 (55.588)
σ_{20}	.835 (.317)	.835 (.317)	.835 (.317)
σ_{21}	−.680 (.261)	−1.320 (.422)	−2.236 (.666)
σ_{u1}^2	151.002 (21.780)	151.002 (21.780)	151.002 (21.780)
σ_{u2}^2	51.250 (3.980)	51.250 (3.980)	51.250 (3.980)
σ_{u3}^2	19.331 (1.644)	19.331 (1.644)	19.331 (1.644)
σ_{u4}^2	9.118 (.817)	9.118 (.817)	9.118 (.817)
σ_{u5}^2	5.850 (.358)	5.850 (.358)	5.850 (.358)
σ_{u6}^2	4.804 (.288)	4.804 (.288)	4.804 (.288)
σ_{u7}^2	4.470 (.333)	4.470 (.333)	4.470 (.333)
σ_{u8}^2	4.363 (.362)	4.363 (.362)	4.363 (.362)
σ_{u9}^2	4.328 (.375)	4.328 (.375)	4.328 (.375)
Fit statistics			
χ^2 (*df*)	165 (42)	165 (42)	165 (42)
RMSEA (CI)	.109 (.092–.127)	.109 (.092–.127)	.109 (.092–.127)
CFI	.915	.915	.915
TLI	.927	.927	.927

Note. Standard errors contained within parentheses unless otherwise noted. Em dashes signify that parameter is not estimated. RMSEA = root mean square error of approximation; CFI = comparative fit index; TLI = Tucker-Lewis index.

model was identical across the different approaches indicating the model-implied trajectories did not vary as a function of using the latent change or latent acceleration approaches. Additionally, several parameter estimates were identical including the intercept mean (μ_0) and variance (σ_0^2), mean and variance of the rate of approach (μ_2 and σ_2^2), covariance between the intercept and rate of approach (σ_{20}), and the residual variances ($\sigma_{u1}^2 - \sigma_{u9}^2$). The factor loadings for x_{1n} and its associated parameters, such as its mean (μ_1) variance (σ_1^2) and covariances with the intercept (σ_{10}) and rate of approach (σ_{21}), were altered due to its rotation (see Zhang et al., 2012). Parameter estimates from the latent change and acceleration models can be rotated back to the traditional exponential growth model to aid interpretation. That is, x_{1n} in the exponential model has a clear interpretation—change from time $t = 0$ to the asymptotic level; however, x_{1n} in the latent change and acceleration model loses that clear interpretation.[2]

Focusing on the traditional approach to fitting the exponential model, the mean of the intercept was 56.395 s representing the predicted reaction time for kindergarten children in this sample. On average, reaction time was predicted to improve 36.696 s to an asymptotic level of 19.699 s. The average rate of approach to the asymptotic level was .430 and is indicative of the shape of the exponential curve. The factor loadings for x_{1n} are very informative in this framework because they are indicative of the rate of approach because the mean of the rate of approach (μ_2) is the only parameter controlling how the factor loadings for x_{1n} change. For example, the factor loading for y_2 (first grade) was .349 and indicates that 35% of the total change to the asymptotic level was gained from kindergarten through first grade, on average. Furthermore, the factor loading for y_9 (eighth grade) was .968 indicating that approximately 97% of the total change to the asymptotic level was gained by eighth grade. Thus, on average, the participants were close to their predicted idealized performance by eighth grade.

There were significant between-child differences in the intercept ($\sigma_0^2 = 192.073$), total change to the lower asymptotic level ($\sigma_1^2 = 138.217$), and rate of approach ($\sigma_2^2 = 0.019$). Thus, children differed in their RT in kindergarten, showed different amounts of improvement in RT, and approached their asymp-

[2]The rotation of x_{1n} from the latent change model to the traditional model in the exponential model is

$$\mu_{1(lcm)} = \mu_{1(lcs)} \cdot \frac{\mu_2 \cdot \exp(-\mu_2)}{1 - \exp(-\mu_2)},$$

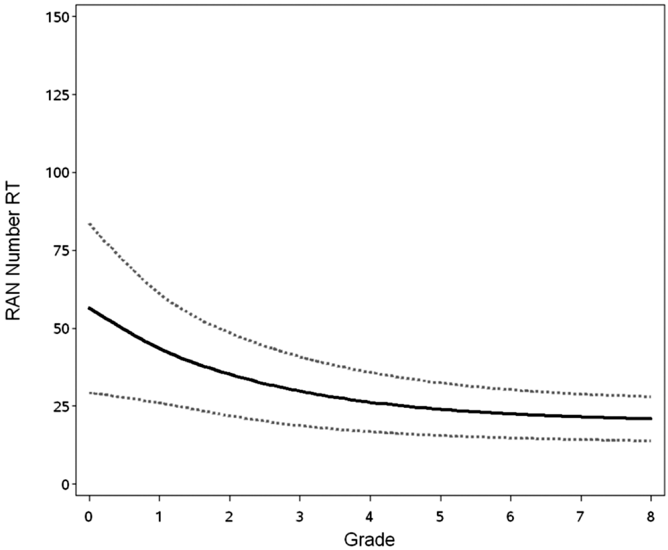
where $\mu_{1(lcm)}$ is the mean of x_{1n} in the traditional exponential model, $\mu_{1(lcs)}$ is the mean of x_{1n} in the latent change exponential model, $\mu_2 \cdot \exp(-\mu_2)$ is the factor loadings from x_{1n} to the latent change scores at the second occasion in the latent change score model, and $1 - \exp(-\mu_2)$ is the factor loading from x_{1n} to the true score at the second occasion in the traditional specification. Additionally, we note that the correlations involving x_{1n} are invariant across frameworks.

totic level at significantly different rates. Furthermore, children who had slower reaction times in kindergarten tended to show greater improvements in their reaction time (correlation between x_{0n} and x_{1n} was $\rho_{10} = -.97$). Children who had slower RT in kindergarten tended to approach their asymptotic levels more quickly (correlation between x_{0n} and x_{2n} was $\rho_{20} = .44$). Finally, children who showed more total improvement tended to approach their asymptotic level more quickly (correlation between x_{1n} and x_{2n} was $\rho_{21} = -.42$). Figure 5A is a plot of the mean predicted trajectory with 95% confidence bound on the between-person differences in the trajectory.

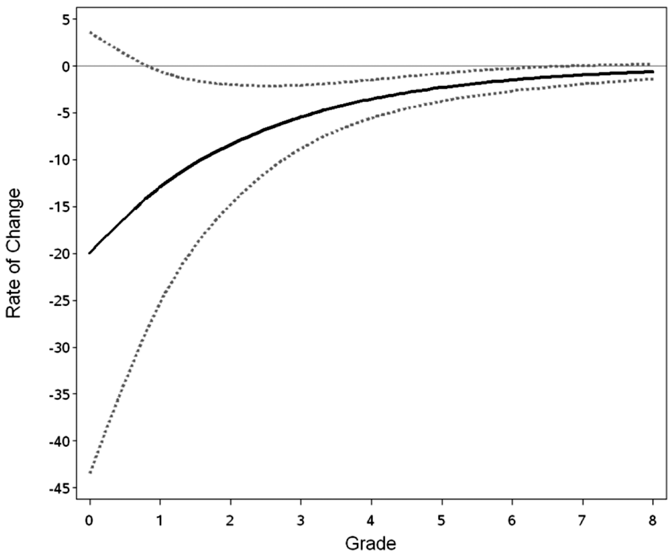
In the latent change and acceleration frameworks the information just presented can be obtained, although some of this information is gained indirectly through transformations. Specifically, the mean of x_{1n}, μ_1, is not the total amount of change to the asymptotic level in the latent change and acceleration frameworks. That is, the scale of x_{1n} in these frameworks prevents direct interpretation of its associated parameter estimates. Thus, parameter estimates associated with x_{1n} should be rotated before interpreted when fitting models within the latent change and acceleration frameworks.

The latent change and acceleration frameworks do provide additional information regarding rates of change, variability in rates of change, acceleration, and variability in acceleration across time and participants that the traditional growth modeling framework does not. Model implied means and variances of the latent change and acceleration scores were requested from the latent acceleration model. From this information we calculated the mean *rate of change* and *acceleration* across time as well as a 95% confidence bound on the between-person differences in each. This information is plotted in Figures 5B and 5C.[3] From Figure 5B, we see that the rate of change gradually approaches 0 as children progress through school and approach their asymptotic RT on the RAN Numbers. Additionally, we note large individual differences in the rate of change in primary school and the magnitude of the individual differences in the rate of change diminishes as children progress through late elementary and middle school. A reference line at a rate of change equal to 0 is included in the plot and from this reference line we can see that some children are not predicted to show changes in their RT after sixth grade. From Figure 5C, we see that the average acceleration, or how quickly the rate of change is changing, gradually diminishes over time as well as the magnitude of the between-person differences

[3]Estimates of the rate of change at the first occasion and estimates of acceleration at the first and second occasion were generated by including latent variables before the first measurement occasion in kindergarten. Including latent variables before the first measurement occasion enabled the inclusion of a latent change score for the kindergarten occasions and latent acceleration scores for the kindergarten and first-grade occasions. The inclusion of these latent variables does not change model fit or the model implied trajectories but provides information regarding rate of change and acceleration at the first two occasions.

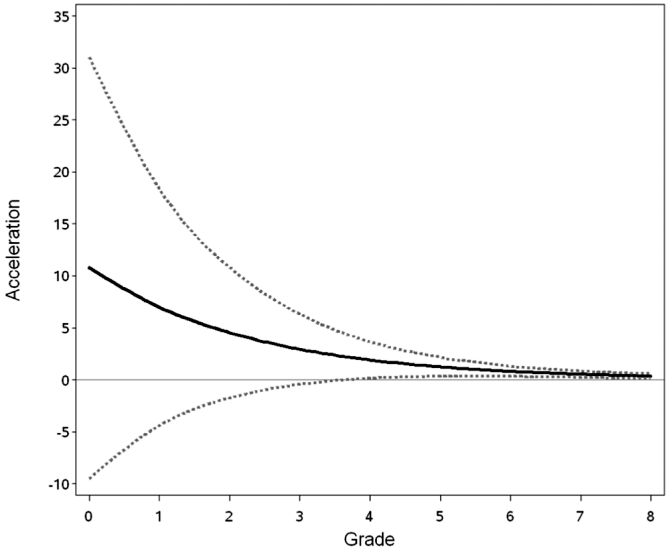


(a)



(b)

FIGURE 5 (A) Predicted mean trajectory with 95% interval on the between-person differences, (B) Predicted rate of change with 95% interval on between-person differences, and (C) Predicted acceleration with 95% interval on between-person differences based on exponential growth curve. *Note.* RAN = rapid automatized naming; RT = response time. (*continued*)



(c)

FIGURE 5 (*Continued*).

in acceleration. By eighth grade, acceleration is near 0 as children approach their asymptotic level on this particular measure of lexical retrieval.

Inclusion of Time-Invariant Covariates

Children's MLD was dummy coded (0 = normative, 1 = MLD), mean centered, and included as a predictor of x_{0n}, x_{1n}, and x_{2n} to evaluate mean differences in these individual parameters between children categorized as normative versus children categorized with an MLD. Mean centering makes the Level 2 intercepts (γ_{00}, γ_{01}, and γ_{02}) equal the expected sample level values and the Level 2 regression coefficients (γ_{10}, γ_{11}, and γ_{12}) remain the expected difference between children with versus without an MLD.

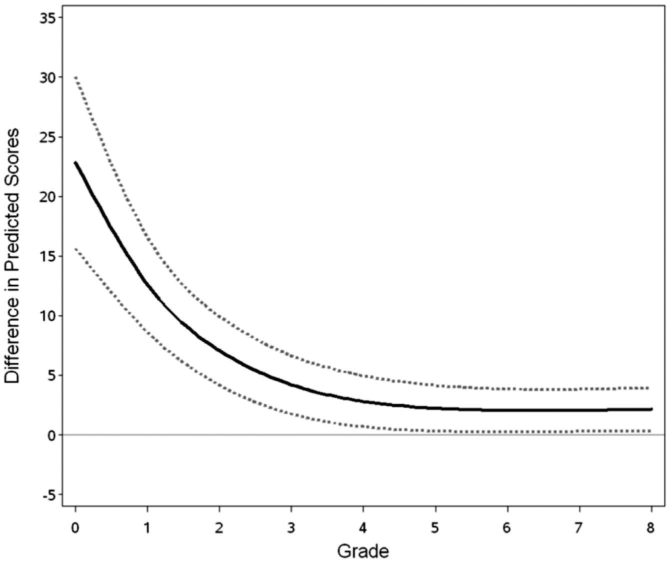
Parameter estimates for these effects are presented in Table 2 for the three different frameworks. The predicted differences in x_{0n} were identical across the three frameworks because the intercept of the exponential model has the same interpretation in each framework—predicted performance when $t = 0$ (kindergarten). Results suggest children with MLD had slower reaction times (by 22.82 s) compared with normative children at kindergarten indicating their fluency with number naming to be delayed, consistent with evidence of deficits in

TABLE 2
Parameter Estimates for the Traditional Exponential Growth Model, Exponential Growth Model Based on Latent Change Scores, and Exponential Growth Model Based on Latent Acceleration Scores With Time-Invariant Covariates

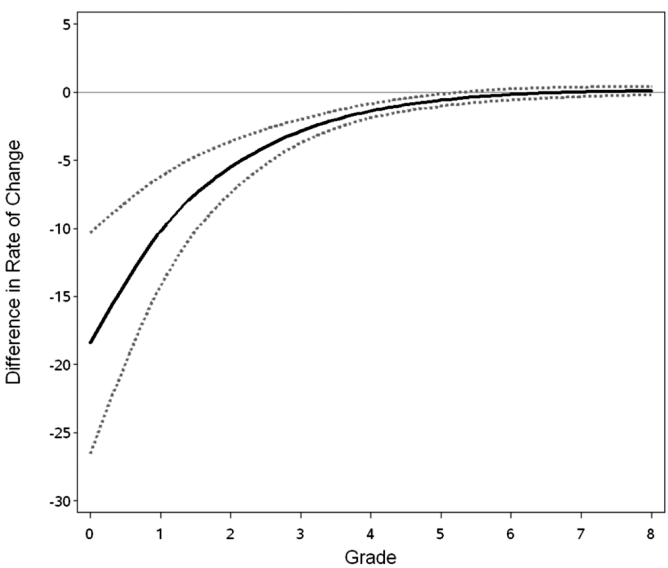
Parameter	*Traditional*	*Latent Change Exponential*	*Latent Acceleration Exponential*
Regression coefficients			
γ_{00}	56.535 (1.111)	56.535 (1.111)	56.535 (1.111)
γ_{10}	22.823 (3.664)	22.823 (3.664)	22.823 (3.664)
γ_{01}	−36.818 (.955)	−46.054 (1.437)	−57.607 (2.173)
γ_{11}	−20.090 (3.204)	−28.418 (4.802)	−39.662 (7.237)
γ_{02}	= 0.000	= 0.000	= 0.000
γ_{12}	.133 (.052)	.133 (.052)	.133 (.052)
Variance/Covariance parameters			
σ_0^2	150.436 (26.078)	150.436 (26.078)	150.436 (26.078)
σ_1^2	106.206 (19.640)	204.613 (44.793)	412.566 (103.077)
σ_2^2	.017 (.005)	.017 (.005)	.017 (.005)
σ_{10}	−122.036 (22.203)	−166.741 (33.536)	−226.196 (50.035)
σ_{20}	.571 (.298)	.571 (.298)	.571 (.298)
σ_{21}	−.456 (.245)	−.988 (.400)	−1.758 (.637)

mapping number names to quantities among children with MLD (e.g., Mazzocco, Feigenson, & Halberda, 2011). The estimated effects of having an MLD on x_{1n} varied over frameworks due to its rotation in the latent change and acceleration models. However, standardized coefficients were invariant across the frameworks indicating the MLD status accounted for the same proportion of variance in x_{1n}. Estimates indicated children with MLD showed greater improvements in reaction time (by 20.09 s) compared with normative children—likely associated with their slower reaction time during kindergarten. Finally, differences in x_{2n}, the rate of approach to the asymptotic level, were identical across the three frameworks and indicated that children with MLD approached their asymptotic level more quickly than normative children. The rate of approach was .133 greater for children with MLD. Thus, it appears that children with MLD reach their optimal performance earlier than normative children. Overall, the effects of the time-invariant predictor were identical over framework even though the information is conveyed with different estimates due to the rotation of x_{1n}.

Indirect effects from the time-invariant covariate were then calculated in each framework. In the traditional growth modeling framework, the total indirect effects from the time-invariant covariate to the true scores translate into the expected differences in RAN Number RT at each occasion. Figure 6A is a plot of this indirect effect along with 95% confidence bound. As seen in this

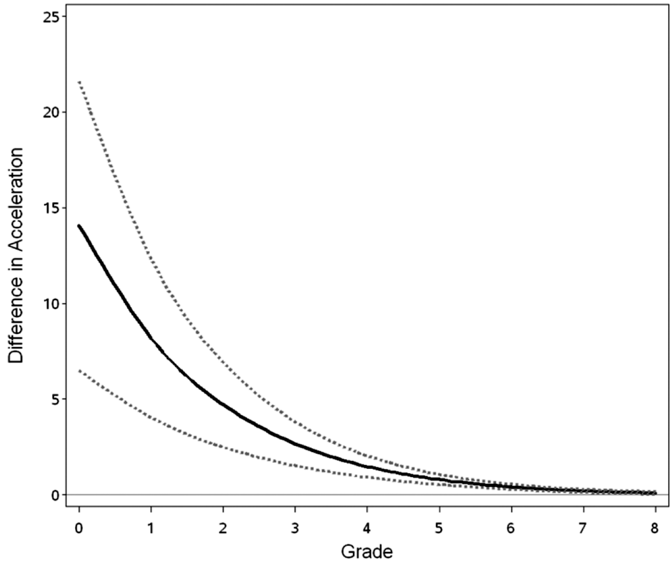


(a)



(b)

FIGURE 6 Differences in RAN Numbers for normative children versus children with a mathematical learning disability in (A) predicted scores, (B) rate of change, (C) acceleration.



(b)

FIGURE 6 (*Continued*).

figure, the difference between normative children and children with MLD in RAN Numbers RT diminish over time; however, they remained statistically significant over the entire observation period (kindergarten through eighth grade). Thus, normative children were predicted to outperform children with MLD in every grade; however, these differences were largest during the early grades and smallest toward the end of the observation period. Thus, examining indirect effects in the traditional latent growth model allows for the examination of expected differences in *status* or *position* (RT) at each measurement occasion and does not examine differences in the *rate of change* or the *acceleration* of the growth trajectory.

Indirect effects were then modeled in the latent change and latent acceleration frameworks. Indirect effects to the true scores can be studied in these frameworks in the same manner and the same conclusions are reached. However, additional indirect effects can be studied. That is, indirect effects from the time-invariant covariate to the latent change scores in the latent change framework and to the latent change and acceleration scores in the latent acceleration framework. Plots of the indirect effects to the latent change scores are contained in Figure 6B. In this figure the difference in the rate of change between the normative children and children with MLD is plotted along with 95% confidence bounds.

Differences in the rate of change between the groups gradually diminished over time and became nonsignificant by sixth grade. Thus, the normative children and children with MLD do differ in the *rate of change* through elementary school but do not differ in the rate of change of this trajectory in junior high school.

Utilizing the latent acceleration framework we additionally examined the effect of the time-invariant covariate on acceleration by examining the total indirect effect from the time-invariant covariate to the latent acceleration scores. The difference in acceleration, along with 95% confidence interval, between the normative children and children with MLD is plotted in Figure 6C. Differences in acceleration diminished over time. However, differences in acceleration were significant across the entire observation period. That is, children with MLD had greater acceleration of their reaction time trajectories from kindergarten through eighth grade compared with normative children. Thus, utilizing the latent change and acceleration frameworks can aid in the interpretation of the effects that time-invariant covariates have on the *rate of change* and *acceleration* of the growth trajectories.

DISCUSSION

In this article, we presented how the latent change and acceleration frameworks can be used to study individual differences in the *rate of change* and *acceleration* in latent growth models. The approach does not change the model-implied trajectories or model fit but is able to provide additional information regarding (a) how the rate of change and acceleration of the growth curve changes over time, (b) individual differences in these aspects of growth, and (c) how time-invariant covariates affect these aspects of the growth. The proposed approach shares similarities with work by Zhang et al. (2012) on growth rate models, where they showed how growth models could be reparameterized to focus on the instantaneous rate of change. The current approach involving the latent change framework can be seen as a discrete analog. However, the current approach allows for the examination of individual differences in acceleration in addition to the rate of change.

The proposed approach is especially useful when modeling change that is nonlinear with respect to time and when multiple parameters simultaneously affect the rate of change (and acceleration). Growth models with nonlinear trajectories are a valuable tool for longitudinal data analysis and have several benefits over the linear growth model as periods of acceleration and deceleration, and asymptotic levels can be studied (Burchinal & Appelbaum, 1991; Grimm et al., 2011). However, nonlinear growth models are not without their limitations. In addition to the lack of focus on the rate of change, a limitation of certain models with nonlinear trajectories, such as the quadratic model, is that model parameters are difficult to interpret (Cudeck & du Toit, 2002). However, by focusing on the rate of change and acceleration and between-person differences therein makes the structure of change and the potential lack of interpretability of model parameters a nonissue.

A third limitation of growth models with nonlinear trajectories is that certain models have parameters that lie outside of the observation period. For example, the logistic growth model has lower and upper asymptotic levels, which lie outside the observation period, and trying to understand individual differences in these parameters may be meaningless because they are never realized. Similarly, b_{1n} in the exponential model (Equation 1) is the predicted individual change from $t = 0$ to the individual's asymptotic level, which lies outside of the observation period. In our illustrative example, participants' changed $\sim$97% of the way to their asymptotic level (loading of x_{1n} at Grade 8), but evaluating total change to the asymptotic level extends beyond the eighth-grade assessment. Thus, examining between-person differences in individual change to their asymptotic level or the asymptotic level itself is less informative than examining between-person differences in change to the final timepoint (eighth grade). Examining the indirect effects from time-invariant covariates to observed scores in nonlinear growth models can provide this information along with the appropriate standard error to evaluate its significance.

A fourth limitation involves varying parameterizations of the same model. As described in the introduction, nonlinear models are often reparameterized to highlight certain aspects of the curve over others. These reparameterizations can make comparisons across studies difficult. Utilizing the latent change and acceleration frameworks make reparameterizations unimportant because the focus is placed on the rate of change and acceleration, which are invariant to reparameterizations.

Limitations of the Proposed Approaches

Fitting nonlinear growth models within the latent change and acceleration frameworks provides additional information about the growth trajectories; however, these approaches are not without their shortcomings. The main limitation is the discrete nature of time. That is, these models cannot be fit to longitudinal data with individually varying timepoints without some adjustment—a limitation shared with growth models fit within the basic structural equation modeling framework. The second limitation is on determining the functional constraint for the factor loading(s) to the first latent change score in the latent acceleration framework. In simple additive models, the factor loading is equal to $\frac{\lambda_{2,las} \cdot \lambda_{1,lcs}}{\lambda_{2,lcs} - \lambda_{1,lcs}}$, where $\lambda_{2,las}$ is the factor loading to the second latent change score in the latent acceleration score model, $\lambda_{1,lcs}$ is the factor loading to the first latent change score in the latent change score model, and $\lambda_{2,lcs}$ is the factor loading to the second latent change score in the latent change score model. However, in more complex models, such as the exponential model fit to the RAN Numbers RT data, this is not the case. In these cases, the constraint must be determined for each model. The third limitation is the rotation of the growth parameters in the latent change and acceleration frameworks as these growth parameters lose their inherent meaning. The fourth limitation is that certain models have first and/or second derivatives that are undefined at certain values of t and this appears to be problematic when fitting these models within the latent change and acceleration models (e.g., $y_{nt} = b_{0n} + b_{1n} \cdot \sqrt{t}$) as model fits are not identical.

Other Considerations

The exponential growth model was discussed throughout the article for illustrative purposes, especially because of the benefits of the proposed approaches when two or more between-person differences affect the rate of change. However, we note that the exponential model is not the only model that may be appropriate for these data. When approaching longitudinal data that are obviously nonlinear with respect to time, there are various approaches to understanding which models are reasonable representations of the individual change trajectories and between-person differences therein. A confirmatory model comparison approach can be taken where various models are tested against the data. An alternative approach is more exploratory and follows the initial approach taken by Meredith and Tisak (1990), whereby the latent basis model is fit and the basis coefficients are examined as well as the residuals to determine an appropriate functional form as well as to determine the number of growth factors needed (see also Grimm, Steele, Ram, & Nesselroade, in press).

The proposed approach is a model-driven approach, where the functional form of change is specified and factor loadings for the latent change and acceleration scores are specified based upon the first and second derivative of the chosen functional form. Other approaches to understanding rates of change and acceleration can be found in Boker (2001), Boker and Nesselroade (2002), and Deboeck (2010), where the first and second derivatives of a time series are first estimated and then a model, often a dynamical systems model, is estimated to understand the relations among the location (displacement), first, and second derivative. These approaches are often applied to intensive longitudinal data at the individual level; however, similar approaches can be applied to shorter multiperson time series and it would be interesting to determine whether these approaches can be married. Additionally, dynamic models similar to those considered by Boker and based upon latent acceleration score models have been proposed by Hamagami and McArdle (2007).

Concluding Remarks

In closing, the proposed approach to studying change highlights two aspects of longitudinal change that are of high interest to researchers—the *rate of change* and *acceleration* and between-person differences in these important aspects of change. These aspects of change are not captured in the traditional approach to latent growth curve modeling with nonlinear trajectories. In addition to aiding the understanding of longitudinal change, the proposed approach can aid in the analysis of multivariate longitudinal data. Latent change score models are often used to understand time-dependent sequences in longitudinal research and the current approach can be adapted to capture changes in the rate of change and acceleration due to time-varying covariates. Focusing on the rate of change and acceleration is essential to understanding developmental processes and the proposed approaches are a step in this direction.

ACKNOWLEDGEMENTS

The authors would like to thank Jack McArdle, John Nesselroade, Nilam Ram, and Jonathan Helm for their helpful comments. Kevin J. Grimm was supported by a National Science Foundation REESE Program Grant DRL-0815787 and the National Center for Research in Early Childhood Education, Institute of Education Sciences, US Department of Education (R305A06021). Michèle Mazzocco was supported by National Institutes of Health award R01-HD34061-01-09. The opinions expressed are those of the authors and do not represent views of the US Department of Education.

REFERENCES

Beal, S. L., & Sheiner, L. B. (1982), Estimating population kinetics. *Critical Reviews in Biomedical Engineering, 8,* 195–222.

Boker, S. M. (2001) Differential structural equation modeling of intraindividual variability. In L. Collins & A. Sayer (Eds.), *New methods for the analysis of change* (pp. 3–28). Washington, DC: American Psychological Association.

Boker, S. M., & Nesselroade, J. R. (2002). A method for modeling the intrinsic dynamics of intraindividual variability: Recovering the parameters of simulated oscillators in multiwave panel data. *Multivariate Behavioral Research, 37,* 127–160.

Browne, M. W. (1993). Structured latent curve models. In C. M. Cuadras & C. R. Rao (Eds.), *Multivariate analysis: Future directions 2* (pp. 171–198). Amsterdam, The Netherlands: North-Holland.

Browne, M., & Du Toit, S. H. C. (1991). Models for learning data. In L. Collins & J. L. Horn (Eds.), *Best methods for the analysis of change* (pp. 47–68). Washington, DC: American Psychological Association.

Burchinal, M., & Appelbaum, M. I. (1991). Estimating individual developmental functions: Methods and their assumptions. *Child Development, 62,* 23–43.

Cudeck, R., & du Toit, S. H. C. (2002). A version of quadratic regression with interpretable parameters. *Multivariate Behavioral Research, 37,* 501–519.

Deboeck, P. R. (2010). Estimating dynamical systems: Derivative estimation hints from Sir Ronald A. Fisher. *Multivariate Behavioral Research, 43,* 725–745.

Denckla, M. B., & Rudel, R. (1974). Rapid "automatized" naming of pictured objects, colors, letters, and numbers by normal children. *Cortex, 10,* 186–202.

Grimm, K. J., & Ram, N. (2009). Nonlinear growth models in Mplus and SAS. *Structural Equation Modeling, 16,* 676–701.

Grimm, K. J., Ram, N., & Hamagami, F. (2011). Nonlinear growth curves in developmental research. *Child Development, 82,* 1357–1371.

Grimm, K. J., Steele, J. S., Ram, N., & Nesselroade, J. R. (in press). Exploratory latent growth models in the structural equation modeling framework. *Structural Equation Modeling*.

Grimm, K. J., & Widaman, K. F. (2010). Residual structures in latent growth curve analysis. *Structural Equation Modeling, 17,* 424–442.

Hamagami, F., & McArdle, J. J. (2007). Dynamic extensions of latent difference score models. In S. M. Boker & M. J. Wenger (Eds.), *Data analytic techniques for dynamical systems* (pp. 47–85). Mahwah, NJ: Erlbaum.

Jolicoeur, P., Pontier, J., Pernin, M.-O., & Sempé, M. (1988). A lifetime asymptotic growth curve for human height. *Biometrics, 44,* 995–1003.

Jöreskog, K. G. (1970). Estimation and testing of simplex models. *British Journal of Mathematical and Statistical Psychology, 23,* 121–145.

Jöreskog, K. G. (1974). Analyzing psychological data by structural analysis of covariance matrices. In R. C. Atkinson, D. H. Krantz, R. D. Luce, & P. Suppas (Eds.), *Contemporary developments in mathematical psychology* (pp. 1–56). San Francisco, CA: Freeman.

Mazzocco, M. M. M., Feigenson, L., & Halberda, J. (2011). Impaired acuity of the approximate number system underlies mathematical learning disability. *Child Development, 82,* 1224–1237.

Mazzocco M. M. M., & Myers, G. F. (2002). Maximizing enrollment efficiency for school based education research. *Journal of Applied Social Psychology, 32,* 1577–1587.

Mazzocco, M. M. M., & Myers, G. F. (2003). Complexities in identifying and defining mathematics learning disability in the primary school age years. *Annals of Dyslexia, 53,* 218–253.

McArdle, J. J. (2001). A latent difference score approach to longitudinal dynamic structural analyses. In R. Cudeck, S. du Toit, & D. Sorbom (Eds.), *Structural equation modeling: Present and future* (pp. 342–380). Lincolnwood, IL: Scientific Software International.

McArdle, J. J. (2009). Latent variable modeling of differences and changes with longitudinal data. *Annual Review of Psychology, 60,* 577–605.

McArdle, J. J., & Epstein, D. (1987). Latent growth curves within developmental structural equation models. *Child Development, 58,* 110–133.

McArdle, J. J., & Hamagami, F. (2001). Linear dynamic analyses of incomplete longitudinal data. In L. Collins & A. Sayer (Eds.), *Methods for the analysis of change* (pp. 137–176). Washington, DC: APA Press.

Meredith, W., & Tisak, J. (1990). Latent curve analysis. *Psychometrika, 55,* 107–122.

Murphy, M. M., Mazzocco, M. M. M., Hanich, L., & Early, M. C. (2007). Cognitive characteristics of children with mathematics learning disability (MLD) vary as a function of the cut-off criterion used to define MLD. *Journal of Learning Disabilities, 40,* 458–478.

Preacher, K. J., & Hancock, G. R. (2012). On interpretable reparameterizations of linear and nonlinear latent curve models. In J. R. Harring & G. R. Hancock (Eds.), *Advances in longitudinal methods in the social and behavioral sciences* (pp. 25–58). Charlotte, NC: Information Age.

Preece, M. A., & Baines, M. J. (1978). A new family of mathematical models describing the human growth curve. *Annals of Human Biology, 5,* 1–24.

Shalev, R. S. (2007). Prevalence of developmental dyscalculia. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 49–60). Baltimore, MD: Paul H. Brookes Publishing CO.

Zhang, Z., McArdle, J. J., & Nesselroade, J. R. (2012). Growth rate models: Emphasizing growth rate analysis through growth curve modeling. *Journal of Applied Statistics, 39,* 1241–1262.